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JANUARY 1958

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

RALPH D. JAMES, *Editor*

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THE TEACHING OF CONCRETE MATHEMATICS

JOHN W. TUKEY, Princeton University and Bell Telephone Laboratories, Inc.

1. Introduction. One syndrome must, from time to time, disturb the sleep of all concerned with the applications of mathematics,—a syndrome never discussed in open meeting, perhaps because of its sensitivity. It seems to be generally agreed that "applied mathematics" is more difficult than "pure mathematics" in requiring more maturity and more years of study before useful results are attained. Today's leaders in "applied mathematics" were mainly trained in "pure mathematics." Yet from the point of view of *research potential and related intellectual ability* the students who *study* in "applied" fields do not compare in strength with those who go into "pure" mathematics! Is this not a paradoxical situation?

One can try to make the situation appear less paradoxical by going further, and asserting that: "Just as today's leaders in the applied fields have come mainly by conversion from the pure, so too will tomorrow's!" (A statement which is undoubtedly true for tomorrow!) But what of the day after tomorrow? Should conversion be inevitable? If it is not, as the writer believes, then the answer must lie in the early training of our students.

Two causes deflect students from the "applied" to the "pure" today:

(1) a feeling among teachers that the "applied" is beneath the "pure,"
(2) a failure to present the "applied" so that it is as intellectually stimulating as the "pure."

Given the temperaments and intellectual orientations of collegiate teachers of mathematics, it is clear that (1) is an inevitable consequence of (2) and that direct (or slanting) attacks on (1) are useless. To improve the situation we must deal with (2), when (1) will, more or less slowly, take care of itself.

How then, may we make "applied" or "concrete" mathematics more stimulating? Many ideas may be needed in the long run, but here are some which appeal strongly to the writer:

(a) we may strive to develop the areas of formulation and approximation, where applied mathematics has failed to heed the admonition "physician, heal thyself."

Success here could give us something of a truly mathematical nature worthy of being taught as applied mathematics. At the best, this is a long-range program (and some may term it visionary)—it is discussed briefly in Section 7.

(b) we may introduce more generality into each stage of the teaching of concrete mathematics—for example, after meeting one or two expansions into eigenfunctions, we may give a nonrigorous introduction to eigenfunction expansions in general.

This program could begin tomorrow, or even today, and needs no detailed spell-

ing out here. It will only succeed, however, if it is focussed on concepts rather than rigor. Usually the physicist or other user will not only omit all rigor in his justified haste to treat a particular point, but he will omit or gloss over many concepts of general interest and help. It is for the mathematician to introduce the student to the majority of the concepts first, letting the rigor wait till its appropriate time. New concepts should be injected into the student as gently as possible!

(c) we may emphasize the study of computational procedures in their own right by discussing their general properties—the mathematics of computation—rather than merely grinding through them.

All of these changes require teaching time and student's time. If we are realistic, we must find time, or at any rate most of it, by saving it elsewhere. Where is time now occupied? With the mechanics of computation, numerical and algebraic. How can it be freed? By reorienting our attitude toward computation—by trying to make it less of a road block.

The key to the immediate attack, then, lies in our attitude and practices concerning computation, taken in a most general sense, and its techniques.

As we succeed with such a program, shifting emphasis from avoidable labor of computation to broader concepts on the one hand and the mathematics of computation on the other, we shall be teaching better 20th-century mathematics—better for mathematicians to teach—and better for students to learn.

2. Attitudes toward computation. Computation may be numerical or "algebraic" where the latter term seems in practice to cover all forms of systematic manipulation which are not merely numerical—polynomials, trigonometric functions, indefinite integration, summation (not summability) of series, tensors and logic, to name a few, all have algebras in the sense of orderly procedures of computation. In their essentials, the practices of these diverse forms of computation are the same. Given input data, one performs more-or-less-or-much-less routine operations with the intent of reaching output results of a predetermined form. Interest centers in the certainty, efficiency, and ease of manipulation of the operations. A certain amount of practice is useful, both to promote understanding (which is not helped appreciably by still more extended practice) and to provide a little facility of manipulation (usually a little suffices).

Numerical computation, through the centuries, has often faced up to reality and made things easier. The use of logarithmic tables, even by those who do not know how to recompute them, and of desk calculators and, now, electronic calculators, even by those who cannot repair them, has been a commonplace. Today the "software" comprising the carefully planned interpretive routines, compilers, and other aspects of automative programming are at least as important to the modern electronic calculator as its "hardware" of tubes, transistors, wires, tapes and the like. When a student or a user begins to use an electronic calculator, we do not ask him to learn all the details of the automatic programming—and surely not to learn why these details were chosen instead of

others. A few students and users will develop slowly into designers or programmers, but their number will be few and their treatment special. Let us look to the analogy in all forms of computation.

Throughout computation, since the usual student will not compute steadily, but rather occasionally, even if he continues to compute all his life, emphasis is best laid on methods which are easy to remember—and, more importantly still, easy to relearn when forgotten. We are really concerned with the continuing capabilities of the student, including those which require supplementation by some re-study when used, rather than with student behavior in a final examination.

There are some students (but how few!) who will go on to compute steadily. They require special training, but their needs should not prejudice the training of the larger student body. We do the specialists no injustice to teach them the easy-to-remember way first, even though it may take 10%, or 20%, or 50%, or even 100% longer than the fanciest method when in steady use. This will not dull their interest in, and appreciation of, the fast, hard-to-remember methods. But if we teach the hard-to-remember method first, the occasional computer will never get to the easy-to-remember method, never have a method he can use when he meets a real problem, and thus never solve the problem.

3. An example from numerical computation. But, some may say, teaching easy-to-remember methods means teaching technique, and ideas and concepts will suffer. This is not so. Let us take Aitken's method of interpolation, [3], [4], as an example. The basic problem is to pass a polynomial through given points (x_1, y_1), (x_2, y_2), (x_3, y_3), and so on. If $P(x)$ with numerical subscripts represents a polynomial passing through the points indicated by the subscripts (thus $P_{124}(x)$ passes through points 1, 2 and 4), and if "$-$" stands for any collection of subscripts other than i or j then

$$[(x - x_i)P_{-j}(x) + (x_j - x)P_{-i}(x)]/(x_j - x_i)$$

passes through all the points corresponding to "$-$" and through points i and j. The argument which shows this is simple, direct, and truly mathematical. Hence we may define

$$(x_j - x_i)P_{-ij}(x) = (x - x_i)P_{-j}(x) + (x_j - x)P_{-i}(x)$$

and starting with

$$P_g(x) = y_g,$$

$$P_{gh}(x) = [(x - x_g)y_h + (x_h - x)y_g]/(x_h - x_g)$$

$$= \frac{y_h - y_g}{x_h - x_g}\,x + \frac{x_h y_g - x_g y_h}{x_h - x_g},$$

obtain all the interpolating polynomials we may desire.

If all we wish is the value of the interpolating polynomial at a given x, then the process reduces itself to successive linear interpolations; thus, for $x=6.1$ and a particular array of x_i and y_i, we have:

i	x_i	x_i-x	$y_i=p_i$	p_{1i}	p_{12i}	p_{123i}	p_{1234i}
1	7	0.9	.84510				
2	5	−1.1	.69897	.77935			
3	8	1.9	.90309	.79391	.78469		
4	4	−2.1	.60206	.77219	.78723	.78580	
5	10	3.9	1.0000	.79863	.78359	.78573	.78578

Every computation here is straightforward linear interpolation or extrapolation. Thus

$$.77935 = \frac{(0.9)(.69897) + (1.1)(.84510)}{2.0},$$

$$.78723 = \frac{(-1.1)(.77219) + (2.1)(.77935)}{1.0},$$

and so on. Undoubtedly this is the easiest of all polynomial interpolation schemes to learn or relearn. (Even though, in the hands of the professional computer, it may be a little slow by comparison with some other schemes, its nearly-iterative and checking features are quite valuable.)

It is easy to learn, yet its teaching is not mainly teaching technique. What has to be taught, in order that its functioning be understood is not technique, but rather (i) that there will be an interpolating polynomial, (ii) that linear interpolation between two equal values returns the same value, (iii) that linear interpolation at an endpoint returns the given value. The algorithm for the interpolating polynomial now follows, and from it the numerical algorithm. After learning a few things of mathematical content (he may even be led to try other operations in place of linear interpolation) the student is equipped to do polynomial interpolation of any order, direct or inverse (for no properties of the spacing of the x_i were used), without the need to recall or look up any coefficients.

The more computation that we can teach in such a form, the better—both for applied mathematics and for pure mathematics.

4. An example from algebraic computation. Formal integration is another example of computation. It points a road we should travel, quite a different road from that just indicated. Some teachers of calculus seem to fear integral tables, apparently feeling that their students should not only be able to develop all the elementary formulas, but should have had to do each several times! What is this but teaching unnecessary technique? In Newton's day these formulas were new and interested mathematicians. Today they are of use, rather than interest. So why should we not strive to make them useful? This means learning how to use integral tables, rather than how to derive them.

Some fear the development of "handbook engineers"—persons who cannot operate without handbooks. This fear is complex; *i.e.*, it has both real and imaginary parts. The ex-student who cannot integrate $\int x^7 dx$ or $\int \sqrt{1-x^2}x dx$ with relative ease would be a discredit to the mathematics department, but need inability to find $\int (1+x^2)^{-3/2} dx$ or $\int \sin^2\theta \cos^4\theta d\theta$, without either a handbook or considerable pain, matter? The writer cannot see that it does. (He himself can work out the transformations, but would rather walk across the hall and borrow an integral table. Does this make him less of a mathematician—or only less of a computer?)

If the time that would otherwise be spent in learning how to derive integration formulas were diverted, not away from "mathematics," but to the introduction of additional mathematical ideas, we should make a great gain. In part, this could be done within the integral tables themselves. Consider the treatment of $\int x^n f(x) dx$ for some relatively simple $f(x)$ for which all such integrals are elementary. The most classical integral tables gave formulas for $n=1$, $n=2$ and usually a reduction formula for lowering n by one or two units. Then came tables which gave the explicit result for $n=1$, $n=2$, $n=3$, and $n=4$ before the user had to resort to the reduction formula. Next perhaps to $n=6$, and so on. The more extended tables are more useful, but some mathematicians find them overelaborate.

Why has no one taken the logical next steps? First, the value of $\int x^n f(x) dx$ will be a finite series. We do have notation, including the use of summation signs, with which to represent finite series. Can we not give the worked-out form for $\int x^n f(x) dx$ for general n in nearly every case?

Would this not be much more useful than a reduction formula? Why should user after user have to go through the same operations to deduce the same finite series from the reduction formula? Not only might we save labor by giving the finite series once and for all, but it is quite likely that we might hint successfully, to students and to users, that generality can mean less work. As mathematicians we should favor such hints.

Second, there is a deeper opportunity. Our integral tables give $\int x^n f(x) dx$, and in some applications we may get such expressions, but in others the user is concerned with $\int P(x) f(x) dx$ where $P(x)$ is a particular polynomial. Once we realize that $\int x^n f(x) dx$ corresponds to a finite series, it follows that $\int P(x) f(x) dx$ is also a finite (double) series. If we invert the order of summation, we shall usually find the answer to be a finite series, each of whose coefficients is of the form $b_{j0}a_0 + \cdots + b_{jm}a_m$, where $P(x) = a_0 + a_1 x + \cdots + a_m x^m$. In more abstract terms, the vector of final coefficients is obtained from the vector u_i of polynomial coefficients by multiplication with a constant matrix b_{ji} with numerical entries. We need not use those fearsome words—but we can tabulate the numerical values of the b_{ji} and explain how to use them. To the extent that students and users make use of tables of b_{ji}, they are being introduced to the practice of matrix computation, and, implicitly, to the idea of a linear transformation.

We could use an integral table, were it rightly constructed, to introduce its users to such important mathematical ideas and notations as matrices, summa-

tion sign technique, and linear transformations. If these things come in as an alternative method—one not taught in class, but acceptable in home work or examination—as an alternative method which *saves work*, they will have by far the greatest chance of penetrating the indifference of the student or user not yet awakened to mathematics.

If a good table of indefinite integrals on this pattern takes 500 handbook-sized pages—with a thumb-index and keys like a book on birds or fishes—should we complain? Or even feel badly? The writer has a waiting, gaping vacancy on his desk for such a volume, which might appropriately be called "Integral Tables for the Occasional Integrator"—as do many of his colleagues. The student need not have to have all 500 pages—we can make up a student's 50-page version, containing the first 30 pages, one page in 7 for the next 70, and one page in 40 thereafter (it would not harm the student or weaken the effect if its paging showed the gaps). But the user would have a place for all 500 pages. If the table were laid out and explained properly, a student who had once learned to use it could relearn readily, as an ex-student, just what he needed in a specific situation. Such a person would not be looked down upon as a "handbook engineer," but rather looked up to as a user of mathematics who was controlling his computational problems, and not letting them control him.

5. Some nonexistent examples. There are other sorts of computation—and most of them lack even the beginnings of the tables and handbooks which would make their use easier, or even easy. A complete catalog would be lengthy, but some examples may be illuminating.

The National Bureau of Standards, with support from the National Science Foundation, has undertaken the preparation of a table of functions. Some think of this as a revision of Jahnke and Emde's table [1], now nearly 50 years old, but others think of it as more nearly the first approximation to a "Numerical Tables for the Occasional Figurer." It may well show a substantial amount of this last aspect, and, to the extent it does, it will tend to make numerical computation less of a bar to the applications of mathematics.

Consider ordinary differential equations. What is "ordinary" about them from the user's standpoint? The writer knows of but one table of solutions [2], and has no reason to be tremendously encouraged about its usefulness. Yet there are a number of possibilities. Our books on intermediate differential equations discuss the reduction of second-order linear equations to standard form—yet, though a number of second-order linear equations have been solved, who has ever seen a table of solutions for such equations after reduction to standard form. Why should there not be such a table?

Our books on elementary differential equations have followed for 60 years the mold of Murray's book ([5], 1897)—a book written but two years after Cantor introduced the union of two sets as a formal operation. Such books contain some relatively widely useful methods of solution, and some special methods which amused 19th-century mathematicians. Today they tend to say a little

more about solutions in series, and, even, numerical solutions. If we had a "Differential Equation Table for the Occasional Integrator," we could condemn the minor techniques to the lumber room, and instead teach vastly more important, suggestive, and stimulating topics such as solution in series and numerical solution.

Modern mathematics devotes a great part of its attention to linear expansions of one sort or another. And many linear expansions are of great practical importance. The process of finding coefficients can almost always be regarded as a process of biorthogonal expansion—yet who has seen even a little table of biorthogonal expansions, to say nothing of the "Biorthogonal Expansion Tables for the Occasional Expander."

Let the reader continue the list.

6. Sources. Where are the better integral tables, the usable differential equation tables, the first biorthogonal expansion tables, and all the others, to come from? The writer does not know, but—there are many members of the Mathematical Association of America who are competent mathematicians, fitted to do original work, yet not stimulated enough by current abstract mathematics to be carrying on important research. If only a small portion of them were to consider the interest and reward associated with trying to make these forms of computation simple, general, *and* easy to learn or relearn, there would be hands enough to do much. (Many of these tasks could well be done cooperatively by substantial groups.)

The successful completion of such tasks would do much to aid the healthy and mutually supporting growth of pure and applied mathematics in America—let us hope that they will be completed.

7. Formulation and approximation. Finally, a word about two areas where we have not explored far enough to see which way we should follow—but which we should clearly attack and exploit—formulation and approximation.

It is agreed that formulation of the problem is usually the most important stage in "applied mathematics," just as insight into what theorem is true and (probably) provable is often the most important stage in "pure mathematics." In each case the formation of new concepts or the refinements of old concepts is likely to be an essential step. Insofar as a concept-former is a philosopher, all mathematicians need to be philosophers (of a very special sort).

The formulation of the problem is of the essence—yet who has studied the problem of formulation, who has tried to explain it to the student? Pólya wrote "How to Solve It"; who will now write "How to Formulate It"? Probably no one person can do it; many must work together, almost all of whom must be mathematicians—though they will usually have other skills as well. Studying the problem of formulation, formulating better and better approximations to it, finding useful concepts for its treatment—these are tasks for "applied mathematicians" skilled in formulation.

It is easy to argue that a book on "How to Formulate It" will be empty

of real intellectual content. It would have been easy to argue 30 years ago that a book on the theory of games would lack intellectual content. In the past 30 years many relevant and useful concepts *have* been formulated in game theory. In the next 30 years many relevant and useful concepts *could* be formulated in formulation theory. Perhaps it is time to start, even if the task may be harder.

If we tried to write "How to Formulate It" today we would strike mainly questions: Where can we find examples to adequately set forth the principles? (What are the principles, anyway?). What kind of exercises can be used as homework? And so on, and on, and on. Yet, if "applied mathematics" is to grow properly, if there is to be something teachable and worthy of the name "applied mathematics," someone must tackle this problem—and eventually there must be developed a technique of wide usefulness and acceptability for teaching. This will not be easy, but it is badly needed.

What of approximation? Why is it paired with formulation? It, too, is a major stage in "applied mathematics," a matter of tactics rather than strategy perhaps, but surely a major stage. Without good approximations we should be lost, yet who knows what concepts are important in approximation? (From a routine mathematics course one would feel that taking the first terms of some series, power or Fourier or perhaps something more complex, was the natural approximation when only a few terms were permissible, yet this is often very wrong! How many connect the $(C, 1)$ summability of most Fourier series with practical approximations?) Yet a reasonable number of concepts have already been isolated, and are to be found by looking in corners. Many more concepts are undoubtedly near the surface. A discussion of approximation from the point of view of concepts and principles rather than labor would undoubtedly bring out much that was worthwhile. Here is another area for the formulators!

Who will begin to study approximation theory, rather than the theory of various specific approximations? Who will try to collect the important concepts in approximation, and try to add to them? What will be the results?

The suggestion of this section is merely this: Just as there is an applied mathematics of games, genetics, and mechanics, so there should be an applied mathematics (at least in terms of concepts, perhaps with techniques and operations) of the applications of mathematics. When there is, mathematicians will be able to teach "the applications of mathematics." At present only individual applications can be taught (and it is not likely to be too good for pure mathematicians to teach applications).

8. Summary. In brief we have said and argued that:

(1) the teaching of any form of computation should be directed (in relatively elementary or general courses) toward the occasional computer rather than the steady computer;

(2) this requires emphasis on simple, easy-to-learn-or-relearn methods;

(3) such emphasis need not mean emphasis on techniques, but rather may, and should, bring in more mathematical ideas than its converse, as exemplified by Aitken's interpolation procedure;

(4) the use of integral tables in teaching elementary calculus should be greatly broadened;

(5) integral tables could be redesigned to bring in the beginnings of matrices, finite summation, *etc.*, as methods of saving labor;

(6) many other forms of table, *e.g.*, solutions of differential equations in standard form, biorthogonal expansions, are badly needed;

(7) there are mathematicians competent to develop such tables who the writer feels would gain enjoyment and satisfaction from their preparation;

(8) a reduction in the labor of computation is the only visible way of finding the time and effort to make the study of computation more rewarding;

(9) there is much to be done in connection with the development of applied mathematics of formulation and approximation.

References

1. E. Jahnke and F. Emde, Funktionentafeln mit Formeln und Kurven, 3rd ed., Leipzig and Berlin, 1938; 4th ed., New York, 1945.

2. E. Kamke, Differentialgleichungen, Lösungsmethoden und Lösungen, 2nd ed., Leipzig, 1943 (Ann Arbor, 1945); 3rd ed., Leipzig, 1944 (New York, 1948) especially pp. 289–636.

3. W. E. Milne, Numerical Calculus, Princeton, 1949.

4. L. M. Milne-Thompson, The Calculus of Finite Differences, London, 1933.

5. D. A. Murray, Introductory Course in Differential Equations, New York, 1897.

ON THE LIMITING EQUILIBRIUM OF n MASSES

D. J. BENNEY, Canterbury University College, New Zealand

1. The general problem. A rigid framework rests on a rough horizontal plane (coefficient of friction μ), being in contact with the plane at the $n+1$ points $A_0, \cdots, A_n$. Let the normal reactions at these points be $W_0, \cdots, W_n$, respectively. A horizontal force P is applied at A_0 making an angle θ with a fixed direction, the magnitude of P being gradually increased until equilibrium is about to be disturbed. The initial displacement of the framework can be represented as a rotation about an instantaneous center I. Our problem is to investigate possible positions for I, and, in particular, to consider whether I may coincide with any of the points A_r. If I coincides with A_r then friction will be limiting at each point A_s, $s \neq r$, while if I is distinct from A_r the friction will be limiting at all points A_s.

We consider two problems of this type [1].

2. (i) Straight rod. A light rigid rod A_0A_n rests on a rough horizontal plane, being in contact with the plane at the $n+1$ equidistant points $A_0, \cdots, A_n$. At each point A_s a particle of weight W is fixed to the rod. The force P is applied at A_0 in a direction making an angle θ with A_nA_0.

Firstly we consider the possibility of the initial position of the instantaneous center I coinciding with A_r. Let X and Y be the components of friction at A_r and let $A_{s-1}A_s = 2a$, $s = 1, \cdots, n$. For equilibrium we have

$$X = P\cos\theta, \tag{2.1}$$

$$Y = P\sin\theta + \mu W(n - 2r), \tag{2.2}$$

$$P\sin\theta = (K/r)\mu W, \tag{2.3}$$

where

$$K = \tfrac{1}{2}[r(r+1) + (n-r)(n-r+1)]. \tag{2.4}$$

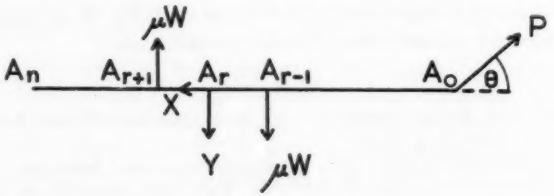

FIG. 1

After eliminating P from (2.1), (2.2), (2.3), we obtain

$$X = \mu W(K/r)\cot\theta, \qquad Y = \mu W[(K/r) + n - 2r].$$

In order that equilibrium should be broken by a rotation about A_r, we require $X^2 + Y^2 < \mu^2 W^2$. This inequality is satisfied provided

$$\operatorname{cosec}^2\theta < \frac{4r[r - n(n-2r)(n-r+1)]}{(2r^2 - 2nr + n^2 + n)^2}. \tag{2.5}$$

Since $\operatorname{cosec}^2\theta \geqq 1$ for real θ, we find that the condition on r is that

$$f(r^2) = r^4 - r^2[n(n+1)+1] + [n(n+1)/2]^2 < 0, \tag{2.6}$$

where r can take the values $0, \cdots, n$.

On solving $f(r^2) = 0$, we obtain the two roots $r = \frac{1}{2}(\sqrt{1+2\lambda} \pm 1)$, where $\lambda = n(n+1)$. Thus, as the difference of the roots is unity, there will, in general, be only one integer between these roots and so rotation will be possible about just one A_r. Hence for a certain range of θ about $\pi/2$, determined by (2.5), we will have I coinciding with A_r, where r is the integer between $\frac{1}{2}(\sqrt{1+2\lambda}-1)$ and $\frac{1}{2}(\sqrt{1+2\lambda}+1)$.

(ii) Exceptional cases. It may happen that $\frac{1}{2}(\sqrt{1+2\lambda}+1)$ is an integer. If this is so, we will have $f(r^2)<0$ only between consecutive integer values and hence no possible A_r can be found. Hence an exceptional case will arise if the equations $r=\frac{1}{2}(\sqrt{1+2\lambda}+1)$, $\lambda=n(n+1)$, possess positive integer solutions.

If $s=\sqrt{1+2\lambda}$ then $r=\frac{1}{2}(s+1)$; and so

$$s^2 = 1 + 2n(n+1) = \tfrac{1}{2}(2n+1)^2 + \tfrac{1}{2} = \tfrac{1}{2}(y^2+1),$$

where $y=2n+1$; *i.e.*, $y^2=2s^2-1$, where $r=\frac{1}{2}(s+1)$, $n=\frac{1}{2}(y-1)$. The general solution of this diophantine equation is

$$s = \frac{1}{2\sqrt{2}}\left[(\sqrt{2}+1)^{2m+1} + (\sqrt{2}-1)^{2m+1}\right],$$

$$y = \frac{1}{2}\left[(\sqrt{2}+1)^{2m+1} - (\sqrt{2}-1)^{2m+1}\right],$$

where m is an integer. It is readily found that the first few integer solutions are

s	y	$n=\frac{1}{2}(y-1)$	$r=\frac{1}{2}(s+1)$
5	7	3	3
29	41	20	15
169	239	119	85
985	1393	696	493

Further exceptional values of n and r can be most easily calculated by using the fact that both s and y satisfy the difference equation $U_p=6U_{p-1}-U_{p-2}$. When n has one of the values 3, 20, 119, 696, *etc.*, it is impossible for equilibrium to be disturbed by rotation about an A_r.

(iii) Simple cases. By using (2.5) it can readily be shown that the following results are valid:

$n=1$: Rotates about A_1 for $\operatorname{cosec}^2\theta<2$, *i.e.*, $\pi/4<\theta<3\pi/4$.
$n=2$: Rotates about A_2 for $\operatorname{cosec}^2\theta<4/3$, *i.e.*, $\pi/3<\theta<2\pi/3$.
$n=3$: Exceptional case; no rotation possible about an A_r.
$n=4$: Rotates about A_3 for $\operatorname{cosec}^2\theta<57/49$.

(iv) Uniform rod. If we put $r/n=\lambda$ in (2.5) and (2.6) and let $n\to\infty$ while λ remains finite, we obtain that $\lambda=1/\sqrt{2}$ if $\theta=\pi/2$. This is the familiar result for a uniform rod.

(v) Locus of the instantaneous center. Let us now suppose that each point A_r slips initially so that I will be distinct from an A_r. Let A_nI make an angle α_n with A_nA_0.

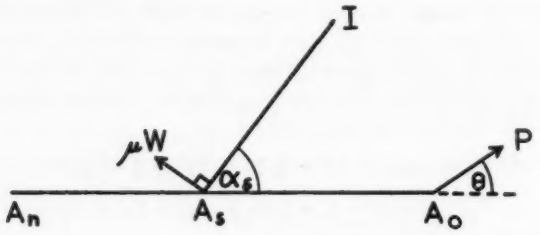

FIG. 2

Taking moments about A_0, we have $\sum_{s=1}^{n} 2as\mu W \cos\alpha_s = 0$, *i.e.*, $\sum_{s=1}^{n} s\cos\alpha_s = 0$. This equation determines the locus of I. Further, this equation shows that I lies on a certain line of force due to collinear point charges s at A_s, $s=1, \cdots, n$; for $\sum_{s=1}^{n} e_s \cos\alpha_s = \text{constant}$ gives the equations of all lines of force due to a set of collinear point charges e_s, $s=1, \cdots, n$. [2].

If each $\alpha_s = \pi/2$, we get a possible position for I so that the locus of I coincides with that line of force having an asymptote perpendicular to the rod.

As a check on our previous work, let us suppose that the curve $\sum_{s=1}^{n} s\cos\alpha_s = 0$ passes through A_r. Then $\alpha_s = \pi$ for $s<r$, $\alpha_s = \beta_r$ for $s=r$, and $\alpha_s = 0$ for $s>r$ must satisfy the equation. Substitution requires that $\cos\beta_r = r - n(n+1)/(2r)$ and, expressing the condition that β_r be real, we obtain the condition

$$r^4 - r^2[n(n+1)+1] + [n(n+1)/2]^2 \leqq 0.$$

This, apart from the possibility of equality, is merely (2.6). When equality occurs we again have the exceptional case and we get $\beta_r = \pi$, $\beta_{r+1} = 0$. This means that the locus of I is the curve separating the lines of force leaving A_r and A_{r+1}, and cutting A_nA_0 at the neutral point N between A_r and A_{r+1}.

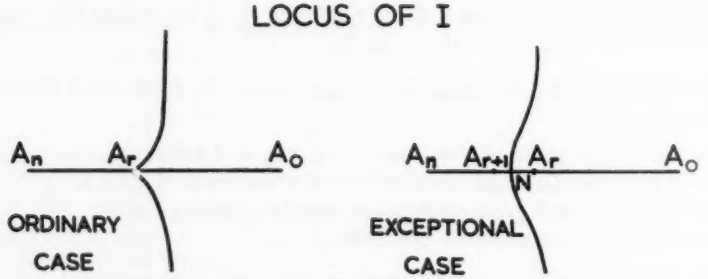

FIG. 3

3. (i) Regular polygon. Let $A_0, \cdots, A_n$ be the vertices of a regular polygon formed by n rigidly jointed light rods. A weight W is placed at each vertex and the polygon rests on a rough horizontal plane (coefficient of friction μ), being in contact with the plane at each vertex. A gradually increasing horizontal force

P is applied at A_0. The direction of P being varied, we wish to consider under what conditions equilibrium will be broken by rotation about a vertex.

Let O be the center of the polygon and suppose that the initial rotation is about A_r, where $r \leqq n/2$ (symmetry about OA_0 will enable us to deduce corresponding results when $r \geqq n/2$). Let the force P make an angle θ with A_rA_0. Let $0 < \theta < \pi$ so that if an initial rotation about A_r is possible, it will be clockwise.

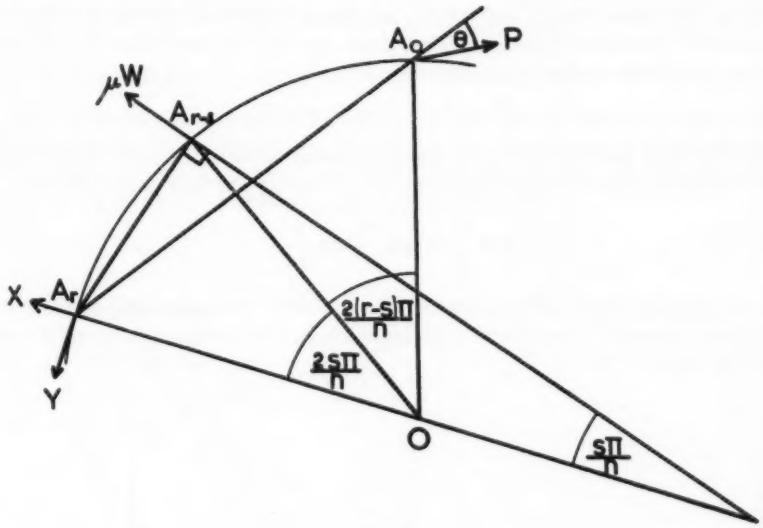

FIG. 4

If X and Y are the friction components at A_r, along and perpendicular to OA_r, we have the following three conditions for equilibrium:

$$(3.1)\qquad X = P\sin\left(\theta + \frac{r\pi}{n}\right) - \mu W \sum_{s=1}^{n-1} \cos\frac{s\pi}{n}, \quad i.e., \quad X = P\sin\left(\theta + \frac{r\pi}{n}\right),$$

$$(3.2)\qquad Y = P\cos\left(\theta + \frac{r\pi}{n}\right) + \mu W \sum_{s=1}^{n-1} \sin\frac{s\pi}{n}, \quad i.e.,$$

$$Y = P\cos\left(\theta + \frac{r\pi}{n}\right) + \mu W \cot\frac{\pi}{2n},$$

$$(3.3)\qquad P\sin\theta\sin\frac{r\pi}{n} = \sum_{s=1}^{n-1} \mu W \sin\frac{s\pi}{n}, \quad i.e., \quad P\sin\theta\sin\frac{r\pi}{n} = \mu W\cot\frac{\pi}{2n}.$$

Using (3.3) to eliminate P from (3.1) and (3.2), we obtain

$$X = \mu W\cot\frac{\pi}{2n}\left(\cot\theta + \cot\frac{r\pi}{n}\right), \qquad Y = \mu W\cot\frac{\pi}{2n}\left(\cot\theta\cot\frac{r\pi}{n}\right).$$

For an initial rotation about A_r we require $X^2+Y^2<\mu^2W^2$. Expressing this condition we find that we require

$$(3.4)\qquad f(\theta)=\cot^2\theta+2\sin\frac{r\pi}{n}\cos\frac{r\pi}{n}\cot\theta+\cos^2\frac{r\pi}{n}-\sin^2\frac{r\pi}{n}\tan^2\frac{\pi}{n}<0,$$

where $0<\theta<\pi$.

If, on the other hand, $-\pi<\theta<0$, we merely need to change the sign of θ and μ in our previous equations. Therefore, the overall condition for an initial rotation about the vertex A_r is that

$$(3.5)\qquad f(\theta)<0,\qquad -\pi<\theta<\pi.$$

(ii) Vertices possible as initial instantaneous centers. Clearly, $f(\theta)>0$ for θ small. Further, $f(\theta)=0$ has imaginary roots if (using the fact that $r\leqq n/2$)

$$(3.6)\qquad \cos^2\frac{r\pi}{n}>\sin\frac{r\pi}{n}\tan\frac{\pi}{2n}.$$

Thus no rotation is possible about A_r if r satisfies the inequality (3.6); but if (3.6) is not satisfied, then I will coincide with A_r for a certain range of θ determined by (3.5).

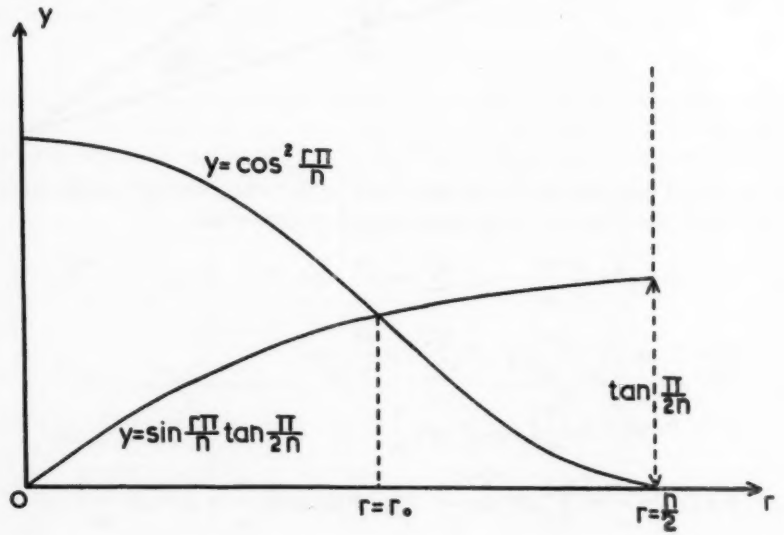

FIG. 5

Consider the graphs of $y=\cos^2(r\pi/n)$ and $y=\sin(r\pi/n)\tan(\pi/2n)$ for r between 0 and $n/2$. If r_0 is the value of r satisfying the equation $\cos^2(r\pi/n)$

$=\sin(r\pi/n)\tan(\pi/2n)$, then, solving, we find that

$$r_0 = (n/\pi)\sin^{-1}(\sqrt{1+\alpha^2}-\alpha),$$

where $\alpha=\frac{1}{2}\tan(\pi/2n)$. This equation has the series solution

$$r_0 = \frac{n}{2} - \sqrt{\frac{n}{2\pi}}\left[1 - \frac{1}{12}\left(\frac{\pi}{2n}\right) + \frac{113}{1440}\left(\frac{\pi}{2n}\right)^2 - \cdots\right].$$

Thus we conclude that I may coincide with A_r if and only if $r>r_0$.

(iii) Uniform circular ring. If we put $r/n=\lambda$ in (3.5) and (3.7) and let $n\to\infty$ while λ remains finite, we obtain $\lambda=\frac{1}{2}$ if $\theta=\pi/2$. This is the well-known result for a uniform circular ring that, if P is applied tangentially, the ring commences to rotate about the diametrically opposite point.

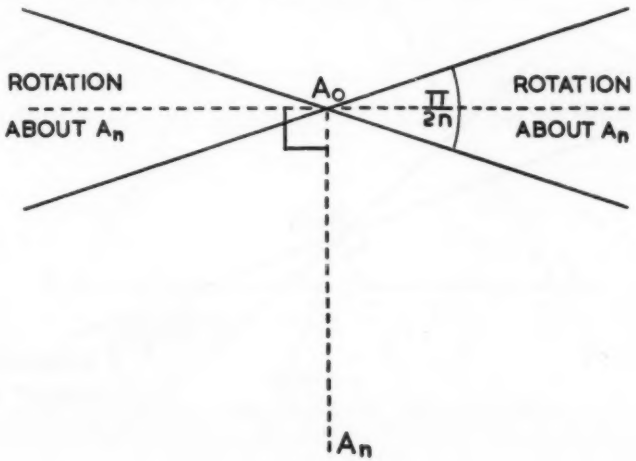

FIG. 6

(iv) Even number of vertices. Let us replace n by $2n$ in the equations. We consider the range of θ for which rotations are possible about A_n, *i.e.*, about the vertex opposite to A_0. Equation (3.5) requires that $\cot^2\theta<\tan^2(\pi/(4n))$. Thus, a rotation about A_n is always possible and will, in fact, occur if

$$\frac{\pi}{2}-\frac{\pi}{4n}<\theta<\frac{\pi}{2}+\frac{\pi}{4n}.$$

For a rod, square, and hexagon, the opposite vertex is the only possible vertex about which a rotation may occur; but for the octagon, neighboring vertices are also possible positions for I.

(v) Odd number of vertices. Let us replace n by $2n+1$ in the equations. We consider the range of θ for which rotations are possible about A_n, *i.e.*, about one of the two vertices opposite A_0. Equation (3.5) requires that

$$\cot\theta[\cot\theta + \sin\pi/(2n+1)] < 0.$$

Thus, a rotation about A_n (and similarly about A_{n+1}) is always possible and will, in fact, occur if

$$\frac{\pi}{2} < \theta < \frac{\pi}{2} + \tan^{-1}\left(\sin\frac{\pi}{2n+1}\right),$$

or if

$$-\frac{\pi}{2} < \theta < -\frac{\pi}{2} + \tan^{-1}\left(\sin\frac{\pi}{2n+1}\right).$$

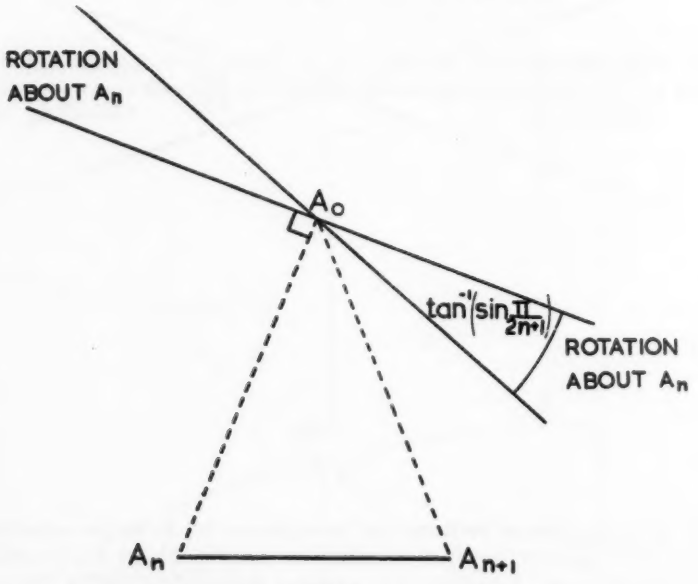

FIG. 7

In particular, for an equilateral triangle, we may have I at A_1 or at A_2. The possibilities are shown in the diagram; when P acts in one of the shaded areas, all three vertices initially slip. In this case, the locus of I consists of the minor arc A_1A_2 of the circumcircle of the triangle, together with the external segment of the line A_1A_2.

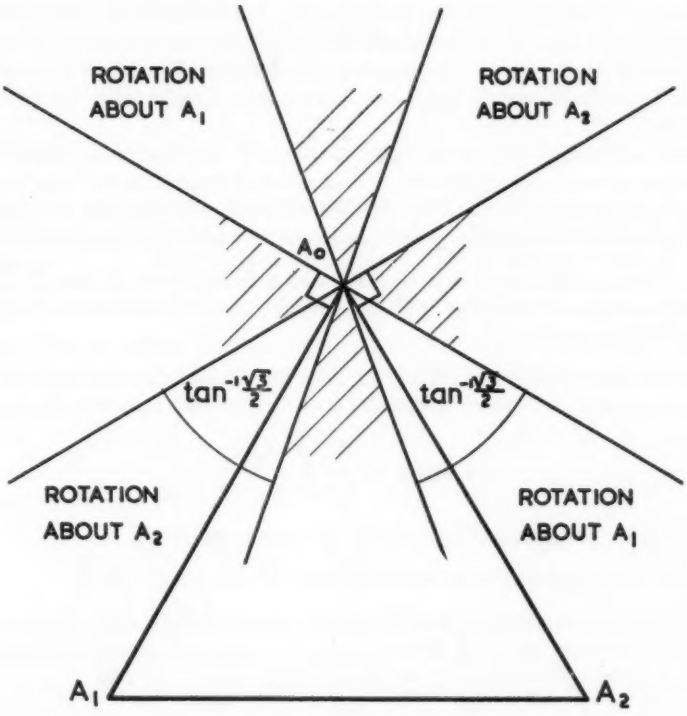

FIG. 8

References

1. E. J. Routh, Analytic Statics, vol. I, Cambridge, 1896, pp. 124, 134.
2. W. R. Smythe, Static and Dynamic Electricity, New York, 1950, pp. 12, 13.

DIVIDED DIFFERENCES IN COMPLEX FUNCTION THEORY

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1. Divided differences have been used to prove the existence and uniqueness of solutions to certain boundary value problems but have been more or less neglected in the usual developments of complex function theory. The purpose of this paper is to show how divided differences may be used to advantage in complex function theory.

The proof of the divided difference theorem is not much more complicated

than that of its special case, Taylor's theorem with remainder. Both proofs are essentially the same. Divided differences, although not mentioned as such, appear in many of the standard complex variables texts in connection with the proof of Taylor's theorem. They also give a very simple form for writing the remainder.

The developments of an analytic function in the forms corresponding to those most common for polynomials and rational functions are also presented here to illustrate further the use of divided differences in complex variables.

The present development presupposes the following five theorems:

1.0. *If $f(z)$ is continuous in a region Ω and γ is any arc in Ω, then $\int_\gamma f(z)dz$ depends only on the end points of γ if and only if $f(z)$ is the derivative of a function which is analytic in Ω.**

1.1. CAUCHY'S INTEGRAL FORMULA.† *If $f(z)$ is analytic in a region Ω and γ is any closed curve in Ω which is homotopic to a point with respect to Ω, then*

$$n(\gamma, z)f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z},$$

$n(\gamma, z)$ being the index (winding number) of γ with respect to z.

1.2. *If $f(z)$ is analytic in a region Ω then $f'(z)$ is analytic in Ω.*

1.3. *If $R(z)$ is a rational function with a zero of order greater than one at ∞ and with its poles at the set of points $\{a_i\}$, $i=1, \cdots, n$, then $\int_\gamma R(z)dz=0$ for every closed curve γ for which $n(\gamma, a_i)=n(\gamma, a_1)$ for $i=2, \cdots, n$.*

This follows from the fact that $R(z)=N(z)/\prod_{i=1}^{n}(z-a_i)^{h_i}$, with $N(z)$ a polynomial of degree less than $h_1+\cdots+h_n-1$, has the partial fraction representation

$$R(z) = \sum_{k=1}^{n}\sum_{i=1}^{h_k}\frac{b_{k,i}}{(z-a_k)^i}$$

with $\lim_{z\to\infty} zR(z)=\sum_{k=1}^{n} b_{k,1}=0$. Hence

$$\int_\gamma R(z)dz = \sum_{k=1}^{n}\int_\gamma \frac{b_{k,1}dz}{z-a_k} = \sum_{k=1}^{n} n(\gamma, a_k)b_{k,1} = \sum_{k=1}^{n} n(\gamma, a_1)b_{k,1} = 0.$$

1.4. REMAINDER THEOREM. *If $F(z)$ is analytic in a region Ω and $a\in\Omega$, then there exists a function $G(z)$ which is analytic in Ω and for which $F(z)=(z-a)G(z)+F(a)$.*

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* Lars V. Ahlfors, Complex Analysis, New York, 1953, the last statement on p. 86.

† The wording is based on that used by F. M. Stewart in an address at the meeting of the Northeastern Section of the Mathematical Association of America, November 24, 1956.

DEFINITION. *The function*

$$G(z) = \begin{cases} \dfrac{F(z) - F(a)}{z - a}, & z \neq a, \\ F'(a), & z = a, \end{cases}$$

usually denoted by $F(a, z)$, *is the first divided difference of* $F(z)$ *with respect to* a *and* z.

If $f(z)$ *is a function which is analytic in* Ω *and* $\{a_i\}$, $i=1, 2, \cdots$, *is a sequence of points in* Ω, *then* $f(a_1, \cdots, a_n, z)$, *the nth divided difference of* $f(z)$ *with respect to* $a_1, \cdots, a_n, z$, *is defined inductively by the recurrence relation*

$$f(a_1, \cdots, a_{n-1}, z) = (z - a_n)f(a_1, \cdots, a_n, z) + f(a_1, \cdots, a_{n-1}, a_n).$$

2. 2.0. THE DIVIDED DIFFERENCE THEOREM. *If* $f(z)$ *is analytic in a region* Ω *and* $\{a_i\}$, $i=1, 2, \cdots$, *is a sequence of points in* Ω, *then, for each* $n>1$,

$$\text{(a)} \qquad f(z) = \sum_{i=1}^{n} f(a_1, \cdots, a_i) \prod_{j=1}^{i-1} (z - a_j) + f(a_1, \cdots, a_n, z) \prod_{j=1}^{n} (z - a_j)$$

with $f(a_1, \cdots, a_n, z)$ *analytic in* Ω *and*

$$\text{(b)} \qquad n(\gamma, z)f(a_1, \cdots, a_n, z) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(\zeta)d\zeta}{(\zeta - z)\prod_{i=1}^{n} (\zeta - a_i)}$$

for every closed curve γ *in* Ω *which is homotopic to a point with respect to* Ω *and for which* $n(\gamma, a_i) = n(\gamma, z)$ *for* $i=1, \cdots, n$.

Proof. The first part of the theorem follows inductively from the definition of $f(a_1, \cdots, a_n, z)$ and from the remainder theorem.

As to the rest of the theorem, 1.1 and 2.0 (a) give

$$n(\gamma, z)f(a_1, \cdots, a_n, z) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(a_1, \cdots, a_n, \zeta)d\zeta}{\zeta - z}$$

$$= \frac{1}{2\pi i}\int_{\gamma} \frac{f(\zeta) - \sum_{i=1}^{n} f(a_1, \cdots, a_i) \prod_{k=1}^{i-1} (\zeta - a_k)d\zeta}{(\zeta - z)\prod_{k=1}^{n} (\zeta - a_k)},$$

while, from 1.3 one has

$$\int_{\gamma} \frac{\sum_{i=1}^{n} f(a_1, \cdots a_i) \prod_{k=1}^{i-1} (\zeta - a_k)d\zeta}{(\zeta - z)\prod_{k=1}^{n} (\zeta - a_k)} = 0.$$

Hence 2.0 (b) holds.

2.1. PERMUTATIONS. *If* $\{b_i\}$, $i=1, \cdots, n$ *is any permutation of* $\{a_i\}$, $i=1, \cdots, n$, *then* $f(b_1, \cdots, b_n) = f(a_1, \cdots, a_n)$.

This follows immediately from the integral representation of the $(n-1)$th divided difference.

2.2. LAGRANGE INTERPOLATION FORMULA. *If $a_i \neq a_j$ for $i \neq j$, then*

$$f(z) = \sum_{i=1}^{n} f(a_i) \prod_{\substack{k=1 \\ k \neq i}}^{n} \frac{z - a_k}{a_i - a_k} + f(a_1, \cdots, a_n, z) \prod_{k=1}^{n} (z - a_k).$$

Proof. If $z = a_i$ for some $i = 1, \cdots, n$, then the theorem is trivially true. Otherwise, denote z by a_{n+1}. Take a closed curve γ which is homotopic to a point with respect to Ω and for which $n(\gamma, a_i) = 1$ for $i = 1, \cdots, n+1$. Pick closed curves γ_i such that $n(\gamma_i, a_j) = \delta_{ij}$, the Kronecker delta, for $i, j = 1, \cdots, n+1$ and for which $\sum_{i=1}^{n+1} \gamma_i$ is homologous to γ with respect to Ω.

Then, since

$$f(a_1, \cdots, a_n, z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)d\zeta}{(\zeta - z) \prod_{k=1}^{n} (\zeta - a_k)},$$

$$f(a_1, \cdots, a_n, a_{n+1}) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)d\zeta}{\prod_{k=1}^{n+1} (\zeta - a_k)} = \frac{1}{2\pi i} \sum_{i=1}^{n+1} \int_{\gamma_i} \frac{f(\zeta)d\zeta}{\prod_{k=1}^{n+1} (\zeta - a_k)}$$

$$= \sum_{i=1}^{n+1} \frac{f(a_i)}{\prod_{\substack{k=1 \\ k \neq i}}^{n+1} (a_i - a_k)} = \frac{f(z)}{\prod_{k=1}^{n} (z - a_k)} + \sum_{i=1}^{n} \frac{f(a_i)}{(a_i - z) \prod_{\substack{k=1 \\ k \neq i}}^{n} (a_i - a_k)}.$$

The Lagrange interpolation formula follows on solving for $f(z)$.

2.3. INTEGRAL AND DERIVATIVE REPRESENTATIONS. *If $a_i = a$ for $i = 1, \cdots, m$, then, for $n \geq m$,*

$$f(a_1, \cdots, a_n) = \frac{f^{(m-1)}(a_m, a_{m+1}, \cdots, a_n)}{(m-1)!}.$$

In particular $f(a_1, \cdots, a_m) = f^{(m-1)}(a)/(m-1)!$ and hence

$$n(\gamma, a) f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)d\zeta}{(\zeta - a)^{n+1}}.$$

This will follow from the following lemmas:

LEMMA 1. *If $F(z)$ and $G(z)$ are analytic in Ω and γ is any closed curve in Ω, then*

$$\int_\gamma F'(z)G(z)dz = -\int_\gamma F(z)G'(z)dz.$$

The proof of Lemma 1 is an immediate consequence of 1.0 since $FG' + F'G = (FG)'$.

Now letting $G(z)=1/(z-a)^m$, one has

LEMMA 2. *If a is any complex number, m and n are any integers, $F(z)$ is analytic in Ω, and γ is any closed curve in Ω not passing through a, then*

(a) $$\int_\gamma \frac{F'(\zeta)d\zeta}{(\zeta-a)^m} = m\int_\gamma \frac{F(\zeta)d\zeta}{(\zeta-a)^{m+1}}.$$

Moreover,

(b) $$\int_\gamma \frac{F^{(n)}(\zeta)d\zeta}{(\zeta-a)^m} = m(m+1)\cdots(m+n-1)\int_\gamma \frac{F(\zeta)d\zeta}{(\zeta-a)^{m+n}},$$

(c) $$n(\gamma,z)F^{(n)}(z) = \frac{1}{2\pi i}\int_\gamma \frac{F^{(n)}(\zeta)d\zeta}{\zeta-z} = \frac{n!}{2\pi i}\int_\gamma \frac{F(\zeta)d\zeta}{(\zeta-z)^{n+1}},$$

(d) $$\frac{d^n}{dz^n}\int_\gamma \frac{F(\zeta)d\zeta}{\zeta-z} = n!\int_\gamma \frac{F(\zeta)d\zeta}{(\zeta-z)^{n+1}}.$$

(b) follows inductively from (a). (c) follows from Cauchy's integral formula and (b) with $m=1$. (d) follows from Cauchy's integral formula and (c).

Proof of 2.3: In (d), let $z=a$, $n=m-1$ and then let

$$F(\zeta) = \frac{f(\zeta)}{(\zeta-z)\prod_{i=m+1}^{n}(\zeta-a_i)}.$$

Then, if $n(\gamma,a_i)=n(\gamma,z)=1$ for $i=1,\cdots,n$,

$$\begin{aligned} f(a_1,\cdots,a_m,a_{m+1},\cdots,a_n,z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{(\zeta-z)(\zeta-a)^m\prod_{i=m+1}^{n}(\zeta-a_i)} \\ &= \frac{1}{(m-1)!2\pi i}\frac{d^{m-1}}{da^{m-1}}\int_\gamma \frac{f(\zeta)d\zeta}{(\zeta-z)(\zeta-a)\prod_{i=m+1}^{n}(\zeta-a_i)} \\ &= \frac{f^{(m-1)}(a_m,a_{m+1},\cdots,a_n,z)}{(m-1)!}. \end{aligned}$$

The rest of the theorem follows by taking $m=n$ and letting $z=a$ in (c).

2.4. TAYLOR'S THEOREM WITH REMAINDER. *If $f(z)$ is analytic in a region Ω and $a\in\Omega$ then, for $n=1,2,\cdots$,*

$$f(z) = f(a)+f'(a)\frac{(z-a)}{1!}+\cdots+f^{(n-1)}(a)\frac{(z-a)^{n-1}}{(n-1)!}+f_n(a,z)(z-a)^n,$$

with $f_n(a,z)$ analytic in Ω and

$$f_n(a,z) = \frac{f^{(n-1)}(a,z)}{(n-1)!} = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{(\zeta-z)(\zeta-a)^n},$$

for every closed curve γ in Ω which is homotopic to a point with respect to Ω and for which $n(\gamma, a) = n(\gamma, z) = 1$.

This is a consequence of 2.0 and 2.3.

2.5. GENERALIZED CAUCHY ESTIMATE. *If $f(z)$ is analytic in a region Ω, $\{a_k\}$, $k = 1, 2, \cdots$, is any sequence in Ω, $\{z : |z-a| \leqq R\}$ is a subset of Ω, $|a-a_k| \neq R$ for any k, and $M = \max_{|z-a|=R} |f(z)|$, then, for $|z-a| < R$ and all n,*

$$|f(a_1, \cdots, a_n, z)| \leqq \frac{MR}{(R - |z-a|) \prod_{k=1}^{n} |R - |a_k - a||}.$$

Moreover, if $\rho < R$ and every limit point of $\{a_k\}$, $k = 1, 2, \cdots$, is contained in the disk $\{z : |z-a| < \rho\}$, then

$$|f(a_1, \cdots, a_n, z)| \leqq \frac{M'R}{(R - |z-a|)(R-\rho)^{n-\nu}}$$

for $|z-a| < R$ and all $n \geqq \nu = \max_{|a_k-a|>\rho} k$, where

$$M' = \frac{M}{\prod_{|a_k-a|>\rho} |R - |a_k - a||}.$$

These estimates follow readily from the integral representation of $f(a_1, \cdots, a_n, z)$ using the circle with center a and radius R for γ.

If, in the divided difference theorem, the second estimate holds for $\rho = R/3$, then if $|z-a| < \rho$, $|z-a_k| < 2\rho$ for $k > \nu$ and hence

$$\left| \prod_{k=1}^{n} (z - a_k) f(a_1, \cdots, a_n, z) \right| \leqq \frac{M'R(R-\rho)^{\nu}}{R - |z-a|} \prod_{k=1}^{n} \left| \frac{z - a_k}{R - \rho} \right|$$

for $n > \nu$. Since all except ν terms of the product are less than 1, one has an infinite series representation of $f(z)$ which is valid for $|z-a| < R/3$.

2.6. UNIQUE FACTORIZATION THEOREM FOR A COMPACT SUBSET OF A REGION. *If $f(z)$ is analytic in a region Ω and if there is a sequence $\{a_i\}$, $i = 1, 2 \cdots$, in Ω for which $f(a_1, \cdots, a_i) = 0$ for all i, then*

(a) *$f(z) = f(a_1, \cdots, a_n, z) \prod_{i=1}^{n} (z - a_i)$ for every n.*

(b) *Moreover, if $\{a_i\}$, $i = 1, 2, \cdots$, has a limit point $a \in \Omega$ then $f(z) \equiv 0$.*

(c) *Consequently, if $f(z) \not\equiv 0$ and Δ is any compact subset of Ω, then either $f(z)$ has no zeros in Δ or there exists an integer n and an ordered set $\{b_i\}$, $i = 1, \cdots, n$, contained in Δ for which*

$$f(z) = \prod_{i=1}^{n} (z - b_i) f(b_1, \cdots, b_n, z)$$

with $f(b_1, \cdots, b_n, z) \not\equiv 0$ in Δ. In particular, if $f(z) \not\equiv 0$, then every zero of $f(z)$ has a finite order and a neighborhood containing no other zeros of $f(z)$.

Proof. (a) follows from 2.0.

As to the proof of (b), one may as well assume to begin with that $\lim_{n\to\infty} a_n = a$. For, since $\{a_n\}$ has a for a limit point, there is a subsequence $\{a_{n_i}\}$, $i=1,2\cdots$, which converges to a. But $f(z) = \prod_{i=1}^{n}(z-a_i)f(a_1, \cdots, a_n, z)$ for $n=1, 2, \cdots$ implies that $f(a_i)=0$ for all i and if $a_i=b$ for k distinct subscripts then $f^{(k-1)}(b)=0$ (*i.e.* the $(k-1)$th divided difference, $f(b, b, \cdots, b)=0$). Hence $f(a_{n_1}, a_{n_2}, \cdots, a_{n_i})=0$ for $i=1, 2, \cdots$.

Now, the argument in the last paragraph of 2.5, shows that $f(z)=0$ for $|z-a|<R/3$ and hence $f^{(k)}(z)=0$ for $|z-a|<R/3$ and all k.

By one of the usual arguments* one may show that $f(z)\equiv 0$ for $z\in\Omega$, completing the proof of (b).

(c) is an immediate consequence of (a) and (b).

2.6. PARTIAL FRACTION REPRESENTATIONS. *If $f(z)$ is meromorphic in Ω and Δ is any compact subset of Ω, then either $f(z)$ is analytic in Δ or there exists a region Ω', an integer n, and an ordered set $\{b_i\}$, $i=1, \cdots, n$, $\subset\Delta\subset\Omega'$ for which*

$$f(z) = \sum_{j=1}^{n} \frac{A_j}{\prod_{k=1}^{j}(z-b_k)} + F(z),$$

with $F(z)$ analytic in Δ. Moreover, if γ is any closed curve in Ω' which is homotopic to a point with respect to Ω' and for which $n(\gamma, b_k)=n(\gamma, z)=1$ for $k=1, \cdots, n$, then

$$A_j = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\prod_{k=1}^{j-1}(\zeta-b_k)d\zeta}{\zeta_i - z},$$

$$F(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z}.$$

Proof. One first notes, from the definition of a pole, that the poles are isolated and that, for any Δ, one can find a region Ω' containing Δ and a $g(z)$ analytic in Ω' such that for $z\in\Omega'$, $g(z)=f(z)\prod_{i=1}^{n}(z-b_i)$ if $z\neq b_i$, $i=1, \cdots, n$.

Then, by the divided difference theorem,

$$g(z) = \sum_{i=1}^{n} g(a_1, \cdots, a_i)\prod_{k=1}^{i-1}(z-a_k) + g(a_1, \cdots, a_n, z)\prod_{k=1}^{n}(z-a_k).$$

Let $a_k=b_{n-k+1}$ for $k=1, \cdots, n$. Then

$$f(z) = \sum_{i=1}^{n} \frac{g(b_n, \cdots, b_{n-i+1})}{\prod_{k=i}^{n}(z-b_{n-k+1})} + g(b_n, \cdots, b_1, z)$$

* See, for example, p. 102 of Ahlfors.

$$= \sum_{i=1}^{n} \frac{g(b_{n-i+1}, \cdots, b_n)}{\prod_{k=1}^{n-i+1}(z - b_k)} + g(b_1, \cdots, b_n, z)$$

$$= \sum_{j=1}^{n} \frac{g(b_j, \cdots, b_n)}{\prod_{k=1}^{j}(z - b_k)} + g(b_1, \cdots, b_n, z).$$

Now, by the divided difference theorem, $g(b_1, \cdots, b_n, z) = F(z)$ is analytic in Δ and

$$g(b_1, \cdots, b_n, z) = \frac{1}{2\pi i}\int_\gamma \frac{g(\zeta)d\zeta}{(\zeta - z)\prod_{k=1}^{n}(\zeta - b_k)} = \frac{1}{2\pi i}\int \frac{f(\zeta)d\zeta}{\zeta - z}.$$

Moreover,

$$A_j = g(b_j, \cdots, b_n) = \frac{1}{2\pi i}\int_\gamma \frac{g(\zeta)d\zeta}{\prod_{k=j}^{n}(\zeta - b_k)} = \frac{1}{2\pi i}\int_\gamma f(\zeta)\prod_{k=1}^{j-1}(\zeta - b_k)d\zeta.$$

Finally, one notes that $\{b_j\}$, $j=1, \cdots, n$, can be represented by an ordered set of distinct points $\{a_j\}$, $j=1, \cdots, m$, where each a_j appears in the original set h_j times and $h_1 + \cdots + h_m = n$, and that

$$f(z) = \sum_{k=1}^{m}\sum_{i=1}^{h_k} \frac{B_{k,i}}{(z - a_k)^i} + F(z),$$

the Laurent expansion in Δ.

MATRIX PROOF OF PASCAL'S THEOREM

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Pascal's theorem states that the points of intersection of the opposite sides of a hexagon inscribed in a conic lie on a line. Many proofs of this celebrated theorem have been given, the most elegant being, perhaps, that of Sylvester [1], but the matrix proof given here has the advantage that it lends itself easily to generalization.

Let A be a real nonsingular symmetric matrix of order $n \times n$, and X and L_i column vectors whose transposes are $X^t = (x_1, \cdots, x_n)$, $L_i^t = (l_{1i}, \cdots, l_{ni})$. Then $L_1L_2^t + L_2L_1^t$ is a symmetric matrix of order $n \times n$ and rank 2. We seek the eigenvectors corresponding to the two nonzero eigenvalues of the system of equations

$$\lambda AX = (L_1L_2^t + L_2L_1^t)X. \tag{1}$$

Setting $X = aA^{-1}L_1 + bA^{-1}L_2$, where a and b are numbers, we find

$$\lambda(aL_1 + bL_2) = (aL_2^tA^{-1}L_1 + bL_2^tA^{-1}L_2)L_1 + (aL_1^tA^{-1}L_1 + bL_1^tA^{-1}L_2)L_2,$$

or

$$aL_1^tA^{-1}L_1 + bL_1^tA^{-1}L_2 = \lambda b, \qquad aL_2^tA^{-1}L_1 + bL_2^tA^{-1}L_2 = \lambda a.$$

Solving we obtain $a = u_2^2$, $b = \pm u_1u_2$, where

$$u_i^2 = L_i^tA^{-1}L_i. \tag{2}$$

Hence the two linearly independent eigenvectors are

$$u_2L_1 \pm u_1L_2. \tag{3}$$

Let A now be of order 3×3, and consider the conic $X^tAX = 0$, in which a hexagon is inscribed. Denote the vertices of the hexagon by p, q, r, s, t, u, and let the equations of the three lines ps, qt, ru be, respectively,

$$L_1^tX = 0, \qquad L_2^tX = 0, \qquad L_3^tX = 0, \tag{4}$$

so that the quadratic forms associated with the pairs of lines ps, qt; qt, ru; ru, ps are $L_1L_2^t + L_2L_1^t$, $L_2L_3^t + L_3L_2^t$, $L_3L_1^t + L_1L_3^t$. For any value of λ

$$X^t(\lambda A - L_1L_2^t - L_2L_1^t)X = 0 \tag{5}$$

is the equation of a conic passing through the points p, s, q, t, and the conic degenerates into a pair of straight lines if λ is an eigenvalue of the equations $\lambda AX = (L_1L_2^t + L_2L_1^t)X$. The corresponding eigenvector is the point of intersection of two lines through the points p, s, q, t. To the eigenvalue $\lambda = 0$ corresponds that eigenvector which is the point of intersection of the lines ps, qt. To the two nonzero eigenvalues correspond eigenvectors which are the points of intersection of the two other pairs of lines through the points p, s, q, t. These two eigenvectors are, from (3), $u_2L_1 \pm u_1L_2$. Similarly for the pairs of lines through the points q, t, r, u and r, u, p, s we obtain, as eigenvectors corresponding to nonzero eigenvalues,

$$u_3L_2 \pm u_2L_3, \qquad u_1L_3 \pm u_3L_1.$$

Note that $u_i^2 = L_i^tA^{-1}L_i = 0$ is the condition for $L_i^tX = 0$ to touch the conic $X^tAX = 0$, and so if the three lines ps, qt, and ru cut the conic, u_1^2, u_2^2, and u_3^2 are all of one sign. Thus it is clear that the set of points

$$u_2L_1 - u_1L_2, \qquad u_3L_2 - u_2L_3, \qquad u_1L_3 - u_3L_1$$

are real, collinear, and lie on the Pascal line of the hexagon.

The generalization, which we now give, of Pascal's theorem to space of three

dimensions differs from that given by Salmon [2], and seems to be more simple and natural. Let

$$L_1^t X = 0, \quad L_2^t X = 0, \quad L_3^t X = 0, \quad L_4^t X = 0 \tag{6}$$

denote four distinct planes, so that the matrix (L_1, L_2, L_3, L_4) is nonsingular and

$$X^t A X = 0 \tag{7}$$

is a quadric, where A is now of order 4×4. Then for any value of λ

$$X^t(\lambda A - L_1 L_2^t - L_2 L_1^t)X = 0 \tag{8}$$

is the equation of a quadric passing through the curves of intersection of the first two planes with the quadric (7). The quadric (8) specializes to a cone when λ is an eigenvalue of the equations

$$\lambda A X = (L_1 L_2^t + L_2 L_1^t)X$$

and the corresponding eigenvector is the vertex of the cone. Six pairs of planes can be obtained from (6) whose common sections are the six edges of the tetrahedron formed by the four planes, and each of the six pairs yields two cones whose vertices, as we see from (3), are the points

$$\begin{array}{lll}
12)\ u_1L_2 - u_2L_1, & 23)\ u_2L_3 - u_3L_2, & 31)\ u_3L_1 - u_1L_3, \\
41)\ u_4L_1 - u_1L_4, & 42)\ u_4L_2 - u_2L_4, & 43)\ u_4L_3 - u_3L_4, \\
12)^*\ u_1L_2 + u_2L_1, & 23)^*\ u_2L_3 + u_3L_2, & 31)^*\ u_3L_1 + u_1L_3, \\
41)^*\ u_4L_1 + u_1L_4, & 42)^*\ u_4L_2 + u_2L_4, & 43)^*\ u_4L_3 + u_3L_4.
\end{array}$$

Note that the points $ik)$ and $ik)^*$ are the same points as $ki)$ and $ki)^*$, respectively. The set of points 12), 23), 31), 41), 42), 43) are coplanar, for the matrix

$$\begin{pmatrix}
u_2 & -u_1 & 0 & 0 \\
0 & u_3 & -u_2 & 0 \\
-u_3 & 0 & u_1 & 0 \\
u_4 & 0 & 0 & -u_1 \\
0 & u_4 & 0 & -u_2 \\
0 & 0 & u_4 & -u_3
\end{pmatrix}
\begin{pmatrix}
L_1^t \\ L_2^t \\ L_3^t \\ L_4^t
\end{pmatrix},$$

being the product of a matrix of rank three and a nonsingular matrix, is itself of rank three. The plane determined by these six points we call the Pascal plane of the original tetrahedron with respect to the quadric (7). For the same reason the set of points 12)*, 13)*, 14)*, 23), 34), 42) are coplanar. Hence in addition to the Pascal plane we obtain the following four planes:

plane P_1 containing the points 12)*, 13)*, 14)*, 23), 34), 42),
plane P_2 containing the points 23)*, 24)*, 21)*, 34), 41), 13),
plane P_3 containing the points 34)*, 31)*, 32)*, 41), 12), 24),
plane P_4 containing the points 41)*, 42)*, 43)*, 12), 23), 31).

The planes P_1, P_2, P_3, P_4 determine a tetrahedron, and it is readily seen that the Pascal plane cuts the six edges of the tetrahedron in the points 12), 23), 31), 41), 42), 43).

If we take the original tetrahedron formed by the four planes in (6) as a tetrahedron of reference, we can give analytical expression to the above results. In this case the matrix (L_1, L_2, L_3, L_4) becomes the unit matrix, and $u_1^2, u_2^2, u_3^2, u_4^2$ are proportional to the principal minors of order 3×3 in the matrix A. The equation of the Pascal plane is

$$u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 = 0,$$

while the planes P_1, P_2, P_3, P_4 are respectively

$$\begin{aligned} -u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 &= 0,\\ u_1x_1 - u_2x_2 + u_3x_3 + u_4x_4 &= 0,\\ u_1x_1 + u_2x_2 - u_3x_3 + u_4x_4 &= 0,\\ u_1x_1 + u_2x_2 + u_3x_3 - u_4x_4 &= 0, \end{aligned}$$

and determine a tetrahedron whose vertices are

$$\begin{aligned} &(-1/u_1, \quad 1/u_2, \quad 1/u_3, \quad 1/u_4),\\ &(1/u_1, -1/u_2, \quad 1/u_3, \quad 1/u_4),\\ &(1/u_1, \quad 1/u_2, -1/u_3, \quad 1/u_4),\\ &(1/u_1, \quad 1/u_2, \quad 1/u_3, -1/u_4). \end{aligned} \tag{9}$$

Provided that each of the four planes (6) intersects the quadric (7), the above planes and points are real.

The form exhibited by the coordinate-sets (9) shows that the following theorem* is true; *The tetrahedron determined by the planes* (6) *belongs to the same desmic system* [3] *as the tetrahedron determined by the planes* P_1, P_2, P_3, P_4.

References

1. J. Sylvester, Instantaneous demonstration of Pascal's theorem by the method of indeterminate coordinates, Phil. Mag. vol. 37, 1850, p. 212.
2. G. Salmon, Analytic Geometry of Three Dimensions, 4th ed., Dublin, 1882, p. 122.
3. R. Hudson, Kummer's Quartic Surface, Cambridge University, 1905, p. 2.

* The author is indebted to the referee for pointing out this theorem.

MATHEMATICAL NOTES

EDITED BY IVAN NIVEN, University of Oregon

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

A REMARK ON LAURENT EXPANSIONS

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Given a single-valued analytic function f, any Laurent expansion $\sum a_n(z-z_0)^n$ of f can be obtained from any other Laurent expansion $\sum b_n(z-z_0)^n$ of f about the same point by means of an elementary device, and furthermore in many cases the sequence $\{a_n-b_n\}$ is easy to describe. For example, there are two Laurent expansions of $(1+z)^{-1}$ about the origin, one of them an ordinary power series, with $a_n-b_n=(-1)^n$, $n=0, \pm 1, \pm 2, \cdots$.

For convenience take $z_0=0$ and suppose that the annulus A of convergence of $\sum a_nz^n$ surrounds the annulus B of convergence of $\sum b_nz^n$, where both series represent a single-valued analytic function which is meromorphic in the closed annulus C between A and B, and let

$$\sum_{p=1}^{n_k} \frac{a_p^{(k)}}{(z-z_k)^p}$$

be the principal part of f at the pole $z_k \in C$, $k=1, \cdots, N$.

PROPOSITION 1. $a_n-b_n=\sum_{k=1}^{N}\sum_{p=1}^{n_k}\binom{-n-1}{p-1}a_p^{(k)}z_k^{-n-p}$.

Proof. Let α and β be positively oriented circles about the origin, lying in A and B respectively. Using the power series development of $1/z^{n-1}$ at z_k one easily computes the residue

$$\sum_{p=1}^{n_k}\binom{-n-1}{p-1}a_p^{(k)}z_k^{-n-p}$$

of $f(z)/z^{n+1}$ at z_k. Hence

$$a_n-b_n=\frac{1}{2\pi i}\int_{\alpha-\beta}\frac{f(z)}{z^{n+1}}\,dz=\sum_{k=1}^{N}\sum_{p=1}^{n_k}\binom{-n-1}{p-1}a_p^{(k)}z_k^{-n-p}.$$

In case all the poles in C are simple the result is $a_n-b_n=\sum_{k=1}^{N}a_1^{(k)}z_k^{-n-1}$, and as a consequence one obtains

PROPOSITION 2. *If C is the common boundary of A and B, a circle of radius r, and if the only singularities of f on C are simple poles at $re^{i\theta_1}, \cdots, re^{i\theta_N}$, where $\theta_1, \cdots, \theta_N$ are commensurate with π, then the sequence $\{r^n(a_n-b_n)\}$ is periodic.*

TWO-SIDED IDEALS OF THE AFFINE NEAR-RING

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The set of all affine transformations $N(F, A)$ of a vector space A over a division ring F has been considered in [2]. It is shown there that N is a near-ring with commutative addition but not a ring. In addition, if (F, A) has finite dimension, the set S of constant transformations forms a maximal two-sided ideal and N/S is isomorphic to the simple ring of linear transformations. It is stated also that S is the unique largest ideal, although the proof of this is incomplete. To investigate the situation further we allow (F, A) to be of arbitrary dimension and determine all two-sided ideals and hence all homomorphic images of $N(F, A)$.

Let $T(F, A)$ denote the ring of all linear transformations of (F, A). For each ordinal $\nu \geqq 0$ let $T_\nu(F, A)$ be the set of all those linear transformations of (F, A) whose range has dimension less than $\aleph_\nu$. Set $T_{-1}=0$. By the word ideal we shall understand a two-sided near-ring ideal, a subset of N satisfying conditions a, b, c on page 518 of [2]. The term ring ideal shall refer to a subset of T which is a two-sided ideal of T in the ordinary ring sense. Our results are:

I. *For each* $\nu \geqq -1$, $T_\nu+S$ *is an ideal of* $N(F, A)$, *and every ideal of* N *has this form, where* $[T_\nu + S = \{\sigma + \tau \mid \sigma \in T_\nu, \tau \in S\}]$.

II. $N/(T_\nu+S)$ *is isomorphic to* T/T_ν.

In particular, I implies that the set of ideals of N forms a well-ordered chain with S the minimal element. If (F, A) has finite dimension then $T_\nu=T$ for all $\nu \geqq 0$, so that $T_\nu+S=N$. Hence S is the only proper ideal of N and thus would certainly be the unique largest ideal as stated in [2].

From II it follows that every homomorphic (nonisomorphic) image of N is actually a ring and in fact isomorphic to a dense ring of linear transformations. For $\nu \geqq 0$ it is known that T/T_ν is a dense ring containing no nonzero transformations of finite rank.

Proof of I. Let the constant transformation $x \to k$ be denoted by σ_k so that $x\sigma_k=k$ for all $x \in A$. Then any affine transformation $\tau'=\tau+\sigma_k$ where τ is linear. Now let $Q>0$ be an ideal of N. Then Q must contain a nonzero transformation which is not linear. For let $\tau \in Q$ be linear and $\neq 0$. Then there exists $k \in A$ such that $k\tau \neq 0$. Then $\sigma_k \in N$ so that $\sigma_k \cdot \tau \in Q$. But $\sigma_k \cdot \tau = \sigma_{k\tau} \neq 0$ and is a constant transformation and certainly nonlinear. Hence Q contains a transformation of the form $\tau+\sigma_k$ where $k \neq 0$, τ is linear ($\tau=0$ is, of course, possible). Since $NQ \leqq Q$ and $0 \in N$, $0 \cdot (\tau+\sigma_k) = \sigma_k \in Q$. Next we show $Q \geqq S$. Let $\sigma_{k'}$ be any constant transformation. Then there exists a linear transformation α such that $k\alpha=k'$. Since $\sigma_k \in Q$ and $0, \alpha \in N$ we have $(0+\sigma_k)\alpha - 0 \cdot \alpha = \sigma_k \cdot \alpha = \sigma_{k\alpha} = \sigma_{k'} \in Q$. Hence $Q \geqq S$.

Now if $\tau+\sigma_k \in Q$ then $(\tau+\sigma_k)-\sigma_k=\tau \in Q$ since Q is an additive group. Let J be the set of all linear transformations in Q. We shall show that J is a ring ideal of T. Certainly J is an additive group since Q is one. J is a left ring ideal

since $TJ \leqq NQ \leqq Q$ and the elements of TJ are linear, making $TJ \leqq J$. It remains to show $JT \leqq J$ and it is again sufficient to show $JT \leqq Q$. Let $\beta \in J$ and $\gamma \in T$. Then $\beta \in Q$ and $\gamma \in N$ so that $(0+\beta)\gamma - 0 \cdot \gamma = \beta\gamma \in Q$.

Hence we have that $Q = J + S$ where J is a ring ideal of T. But all ring ideals of T have the form T_ν ([1], p. 198), so that $Q = T_\nu + S$ for some $\nu \geqq -1$.

To complete the proof we must show that $Q = J + S$ is an ideal whenever J is a ring ideal of T. Clearly Q is an additive group. Let $\tau + \sigma_a \in N$ ($\tau \in T$) and $\alpha + \sigma_b \in Q$ ($\alpha \in J$). A direct computation yields $(\tau + \sigma_a)(\alpha + \sigma_b) = \tau\alpha + \sigma_{a\alpha} + \sigma_b \in J + S$. Hence $NQ \leqq Q$. To verify condition (c) for Q, let $\tau + \sigma_a$, $\gamma + \sigma_c \in N$ and $\alpha + \sigma_b \in Q$. Then

$$[(\tau + \sigma_a) + (\alpha + \sigma_b)](\gamma + \sigma_c) - (\tau + \sigma_a)(\gamma + \sigma_c) = \alpha\gamma + \sigma_{b\gamma} \in J + S.$$

The last computation can be achieved by applying the previous rule twice.

Proof of II. Consider the mapping $\beta: N = T + S \to T/T_\nu$ defined by $\tau'\beta = (\tau + \sigma_k)\beta = \tau + T_\nu$. This is obviously a near-ring homomorphism of N onto the ring T/T_ν. The kernel of β is the ideal $T_\nu + S$. Since the homomorphism theorem holds for near-rings the conclusion follows.

References

1. R. Baer, Linear Algebra and Projective Geometry, New York, 1952.
2. D. W. Blackett, The near-ring of affine transformations, Proc. Amer. Math. Soc., vol. 7, 1956, pp. 517–519.

ESSENTIAL SIMILARITY: A COUNTEREXAMPLE

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Let G be a group under the operations $x * y$ and $x \# y$. The groups $G(*)$ and $G(\#)$ are called *essentially similar* (*ES*) if there exists c in G such that $x * y \equiv x \# c \# y$. This concept was considered by D. Ellis,* who formulated the following

Conjecture. *If* $G(+, *)$ *and* $G(+, \#)$ *are s-fields, then* (*ES*) *subsists.*

We present the following counterexample. Let the additive group $G(+)$ be the direct sum of the additive group of rational numbers with itself; thus G consists of ordered pairs (r_1, r_2) of rational numbers. Let m, n be distinct positive integers which are not perfect squares and define two multiplications on G by

$$(x_1, x_2) * (y_1, y_2) = (x_1y_1 + mx_2y_2, x_1y_2 + x_2y_1),$$
$$(x_1, x_2) \# (y_1, y_2) = (x_1y_1 + xn_2y_2, x_1y_2 + x_2y_1).$$

Then $G(+, *)$ and $G(+, \#)$ are fields since they are isomorphic to the quadratic number fields $R(\sqrt{m})$ and $R(\sqrt{n})$ under the maps $(x_1, x_2) \to x_1 + x_2\sqrt{m}$ and $(x_1, x_2) \to x_1 + x_2\sqrt{n}$, respectively.

* D. Ellis, Cross-associativity and essential similarity, this Monthly, vol. 60, 1953, pp. 545–546.

If we set $x=(x_1, x_2)$, $y=(y_1, y_2)$, and $c=(c_1, c_2)$, we find by an easy calculation that c obeys the condition of (ES) if and only if the equations

$$x_1y_1 + mx_2y_2 = x_1(c_1y_1 + nc_2y_2) + nx_2(c_1y_2 + c_2y_1), \tag{1}$$

$$x_1y_2 + x_2y_1 = x_1(c_1y_2 + c_2y_1) + x_2(c_1y_1 + nc_2y_2) \tag{2}$$

hold for all rational x_1, x_2, y_1, y_2 such that $(x_1, x_2)\neq(0, 0)$ and $(y_1, y_2)\neq(0, 0)$. Let $x_1=y_1=1$ and $x_2=y_2=0$ to deduce $c_1=1$ from (1); then set $x_2=y_2=1$ and $x_1=y_1=0$ to deduce $m=n$ from (1). Since we originally assumed $m\neq n$, we have shown that no c obeying the condition of (ES) exists.

I am indebted to the referee for pointing out (a) that nonidentical group multiplications with the same identity element cannot be (ES) and (b) that a counterexample in which multiplications have different identity elements can be obtained from the above by the single change $(x_1, x_2)\ \#\ (y_1, y_2)=(x_1y_2+x_2y_1, nx_1y_1+x_2y_2)$.

A DECOMPOSITION THEOREM FOR THE INTEGERS MODULO q*

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During an investigation of finite fields which will be reported elsewhere (a preliminary version has appeared as an internal Lincoln Laboratory document), it was observed that not only is it true that a and a^r are of the same order for every element a of a finite field F with $q+1$ elements when r is prime to q but that whenever a and b are elements of F of the same order there exists r prime to q for which $a^r=b$. It is not difficult to see that the second statement is a direct consequence of the following result. Let q be an arbitrary positive integer and let m be any integer. Then there exist a divisor d of q and a number r prime to q such that $m\equiv dr \pmod q$. If we observe now that d is uniquely determined by m and q we are led to the following

THEOREM. *Let q be a positive integer and let S be the ring of least positive residues modulo q. Let G be the multiplicative group of regular elements of S. Then the collection $\{dG\}_{d|q}$ is a decomposition of S. That is, $S=\bigcup_{d|q} dG$ and $d_1G\cap d_2G=\phi$ if $d_1\neq d_2$.*

LEMMA. *Let $q'|q$ and suppose $(r', q')=1$. Then there is an r in G such that $r\equiv r' \pmod{q'}$.*

Proof. Let d denote the product of the distinct prime factors of q which do not divide q'. Then $(d, q')=1$ and so $\{r'+aq'\}$, $a=0, \cdots, d-1$, is a complete residue system modulo d. Hence for some a_0, $r'+a_0q'\equiv 1 \pmod d$. If the prime p divides $(q, r'+a_0q')$, then $p\nmid q'$ for otherwise $p|(r', q')$. Thus $p|d$ which implies $p|1$ and so $r=r'+a_0q'$ has the required properties.

* The research in this document was supported jointly by the Army, Navy and Air Force under contract with the Massachusetts Institute of Technology.

Proof of the theorem. For each divisor d of q let A_d denote the set of all those a in S for which $(a, q) = d$. Clearly the collection $\{A_d\}_{d|q}$ is a decomposition of S. Since $dG \subset A_d$ is apparent, the proof is complete if $A_d \subset dG$. Let $a \in A_d$ and let $q' = q/d$. Then $(a/d, q') = 1$ and it follows from the lemma that r in G can be found for which $a/d \equiv r \pmod{q'}$. Hence $a \equiv dr \pmod{q}$ which in S means that $a = dr$.

COROLLARY 1. *Let H be a cyclic group of finite order q and for h in H let $\theta(h)$ denote the order of h. Let S denote the ring of least positive residues modulo q. Let $h \in H$ and let k be any member of S for which $q/(k, q) = \theta(h)$. Then there exists a generator g of H for which $g^k = h$.*

Proof. Let f be any generator of H. Then $f^j = h$ for some j with $\theta(h) = \theta(f^j) = q/(j, q)$ and so $(j, q) = (k, q)$. It follows from the theorem that $k = dr$, $j = dr_1$ where $d \mid q$ and r, r_1 are in G. Then $g = f^{r_1 r^{-1}}$ has the required property.

COROLLARY 2. *Let H be a cyclic group of finite order q and let g and h be two elements of H of the same order. Then there exists r prime to q for which $g^r = h$.*

Proof. Let $d = q/\theta(g)$. By Corollary 1 there exist generators f_1 and f_2 of H such that $f_1^d = g$ and $f_2^d = h$. Then $f_2 = f_1^r$ for some r prime to q and so $g^r = h$.

AN n-LINE PROPERTY

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Using the notation and method of the article *A chain of circles associated with the n-line*,* the equation of the Miquel circle with center $C_{n1/f}$ passing through $K_{n1/f}$ is

$$\mathbf{z\bar{z}(n!/f)} - z \sum k_1 \mathbf{1}^{2n-s}\mathbf{(n!/1f)} - \bar{z}(-1)^n \mathbf{n!/f} \sum k_1 \mathbf{n!/1f(n!/1f)}$$
$$= (-1)^n \mathbf{n!/f} \sum k_1 k_2 \mathbf{n!/12f(n!/12f)}\{1^{2n-s} - 2^{2n-s}\}\{1^2 + 2^2\}.$$

The equation of the Miquel circle with center at $C_{n1/g}$ may be written down by replacing **f** by **g** in the above equation. Multiplying the first equation by $\mathbf{(1f)(2f)(3f)} \cdots \mathbf{(nf)}$ and the second by $\mathbf{(1g)(2g)(3g)} \cdots \mathbf{(ng)}$ and subtracting, we find the radical axis of these two circles to be

$$z \sum k_1 \mathbf{1}^{2n-s}\mathbf{(n!/1)} + \bar{z}(-1)^n\{\mathbf{n!/fg}\}^2 \sum k_1 \mathbf{1(n!/1)}$$
$$= (-1)^{n-1}\mathbf{n!/fg} \sum k_1 k_2 \mathbf{n!/12fg(n!/12)}\{1^{2n-s} - 2^{2n-s}\}\{1^2 + 2^2\}\{1^2 2^2 - \mathbf{f^2 g^2}\}.$$

The radical axis of the pair of Miquel circles with centers at $C_{n1/f}$ and $C_{n1/h}$ may be written down by replacing **g** by **h** in the above equation. Multiplying the first equation by $\mathbf{g^2}$ and the second by $\mathbf{h^2}$ and subtracting, we find that the radical center of the three circles is

* This MONTHLY, vol. 63, 1956, p. 306.

$$\frac{(-1)^n \mathbf{n}! \sum k_1 k_2 \mathbf{n}!/12(\mathbf{n}!/12)\{1^{2n-8} - 2^{2n-8}\}\{1^2 + 2^2\}}{\sum k_1 1^{2n-5}(\mathbf{n}!/1)}.$$

Since this is independent of **f**, **g** and **h**, it follows that

THEOREM I. *The n Miquel circles of the $(n-1)$-lines obtained by omitting in turn $l_1, l_2, \cdots, l_n$ from the n-line have a common radical center.*

Let us denote this expression by z_1, the radical center of the "new circles" (p. 312 of the article) by z_2, and the center of the Morley (centric) circle by z_3. If these three points are collinear,

$$(z_1 - z_3)(\bar{z}_2 - \bar{z}_3) = (z_2 - z_3)(\bar{z}_1 - \bar{z}_3).$$

Making use of the fact that $(\mathbf{n}!)(\mathbf{n}!/\mathbf{fg}) = (\mathbf{n}!/\mathbf{f})(\mathbf{n}!/\mathbf{g})(\mathbf{fg})$, it is not hard to show that

$$z_1 - z_3 = \frac{(-1)^n\{\mathbf{n}!\}^2 \sum k_1 1^3(\mathbf{n}!/1) \cdot \sum k_1 1^{2n-9}(\mathbf{n}!/1)}{(\mathbf{n}!) \sum k_1 1^{2n-5}(\mathbf{n}!/1)},$$

$$\bar{z}_1 - \bar{z}_3 = \frac{-\sum k_1 1^5(\mathbf{n}!/1) \cdot \sum k_1 1^{2n-7}(\mathbf{n}!/1)}{(\mathbf{n}!) \sum k_1 1(\mathbf{n}!/1)},$$

$$z_2 - z_3 = \frac{(-1)^n\{\mathbf{n}!\}^2 \sum k_1 1(\mathbf{n}!/1) \cdot \sum k_1 1^{2n-9}(\mathbf{n}!/1)}{(\mathbf{n}!) \sum k_1 1^{2n-7}(\mathbf{n}!/1)},$$

$$\bar{z}_2 - \bar{z}_3 = \frac{-\sum k_1 1^5(\mathbf{n}!/1) \cdot \sum k_1 1^{2n-5}(\mathbf{n}!/1)}{(\mathbf{n}!) \sum k_1 1^3(\mathbf{n}!/1)}.$$

The required condition follows immediately. Thus

THEOREM II. *The radical center of the n Miquel circles and the radical center of the n new circles are collinear with the center of the Morley (centric) circle of the n-line.*

THE MAPPING WHICH TAKES EACH ELEMENT OF A GROUP ONTO ITS *n*th POWER

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In an Abelian group the mapping $\bar{n}$ of each element g to g^n is an endomorphism. We consider here the converse question whether a group is Abelian when $\bar{n}$ is an endomorphism.

Let G denote the group with elements a, b, g, *etc.*, and for each integer n let $G(n)$ denote the subgroup of G generated by all nth powers g^n of G; let $G\{n\}$ denote the subgroup of G generated by all elements whose orders divide n.

The results can then be stated as follows:

PROPOSITION. 1) *Let G be a group in which the mapping $\bar{n}$ is an endomorphism, then $G/G\{n^2-n\}$ is Abelian;*

2) *If $\bar{n}$ is an automorphism then $G/G\{n-1\}$ is Abelian;*

3) *If G has no elements whose orders divide n^2-n or if G has no elements whose orders divide $n-1$ when $\bar{n}$ is an automorphism, then G is Abelian.*

It should be noted that if G is the direct product of two groups A and B, where $A(n-1)=(1)$ and $B(n)=(1)$, then $\bar{n}$ leaves A elementwise fixed and maps B into (1). Hence any group of this type admits $\bar{n}$ as an endomorphism, and some such restriction as in 3) is necessary if G is to be Abelian.

The proof of the proposition is as follows. Since $\bar{n}$ is an endomorphism $a^n b^n=(ab)^n$. If a is cancelled on the left and b on the right, then $a^{n-1}b^{n-1}=(ba)^{n-1}$. It follows that $b^{1-n}a^{1-n}=(ba)^{1-n}$ and $\overline{1-n}$ is an endomorphism (*cf.* [1]).

Then $(aa^{n-1}b^n)(a^{1-n}b^{-n}b)=(ab)^n(ab)^{1-n}=ab$, and $1=a^{n-1}b^n a^{1-n}b^{-n}$, or $a^{n-1}b^n=b^n a^{n-1}$. This means that nth powers commute with $(n-1)$st powers, whence $G(n^2-n)$ is Abelian (*cf.* [2] p. 29 Ex. 4).

Now the product of the two endomorphisms $\bar{n}$ by $\overline{1-n}$ is an endomorphism of G onto the Abelian group $G(n^2-n)$ with kernel $G\{n^2-n\}$. This proves the first statement of the proposition.

Statement 2) follows from the fact that when $\bar{n}$ is an automorphism, every element is an nth power, and therefore the equation $a^{n-1}b^n=b^n a^{n-1}$ implies that $G(n-1)$ is in the center of the group. It follows that $\overline{1-n}$ is an endomorphism, mapping G onto the Abelian group $G(n-1)$ with kernel $G\{n-1\}$.

Statement 3) follows immediately from 1) and 2).

We are indebted to the referee for the references to the literature.

References

1. J. W. Young, On the holomorphisms of a group, Trans. Amer. Math. Soc., vol. 3, 1902, pp. 186–191.

2. Hans Zassenhaus, Group Theory (English Edition), New York, 1949.

AN ALGORITHM FOR THE EVALUATION OF FINITE TRIGONOMETRIC SERIES

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The algorithm described below enables one to obtain the simultaneous numerical evaluation of $C=\sum_0^N a_k \cos kx$ and $S=\sum_1^N a_k \sin kx$ for given a_k, $\cos x$, and $\sin x$. Tables for $\sin kx$ and $\cos kx$ are not needed and only about N multiplications and about $2N$ additions or subtractions are required, so the method may be of interest to programmers of digital computers.

The algorithm is defined by

$$U_{N+2}=U_{N+1}=0;$$

$$U_k=a_k+2\cos xU_{k+1}-U_{k+2}, \qquad k=N, N-1, \cdots, 1.$$

$$C=a_0+U_1\cos x-U_2, \qquad S=U_1\sin x.$$

To establish this result, consider

$$V_k = \sum_{j=k}^{N} a_j \sin (j - k + 1)x; \qquad k = 1, \cdots, N,$$

$$V_{N+1} = V_{N+2} = 0.$$

Then

$$a_k \sin x + 2 \cos x V_{k+1} - V_{k+2}$$

$$= a_k \sin x + \sum_{j=k+1}^{N} a_j [2 \cos x \sin (j - k)x - \sin (j - k - 1)x]$$

$$= a_k \sin x + \sum_{j=k+1}^{N} a_j \sin (j - k + 1)x = V_k,$$

whence $V_k = U_k \sin x$ and, in particular, $S = V_1 = U_1 \sin x$. Furthermore

$$a_0 \sin x + V_1 \cos x - V_2 = a_0 \sin x + \sum_{j=1}^{N} a_j [\cos x \sin jx - \sin (j - 1)x]$$

$$= a_0 \sin x + \sum_{j=1}^{N} a_j \cos jx \sin x = C \sin x,$$

whence $C = a_0 + U_1 \cos x - U_2$.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

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A DIRECT PROOF FOR THE LEAST SQUARES SOLUTION OF A SET OF CONDITION EQUATIONS

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The problem of finding the solution of a set of m independent "condition" equations, linear in the n variables $v_1, \cdots, v_n$, $n > m$

$$\sum_{j=1}^{n} a_{ij} v_j - a_{i0} = 0, \qquad i = 1, \cdots, m \tag{1}$$

such that $\sum_1^n v_j^2$ be a minimum is generally solved, following Lagrange, by minimizing instead, an equivalent function involving the so-called Lagrangian multipliers.

The following approach seems more direct, and generalizes a basic theorem in analytic geometry to n dimensions.

Multiplying in turn each of equations (1) by one of the m parameters

$k_1, \cdots, k_m$, and adding, results in the single linear equation

$$\sum_{j=1}^{n} \sum_{i=1}^{m} k_i a_{ij} v_j - \sum_{i=1}^{m} k_i a_{i0} = 0 \tag{2}$$

which contains all points $P^*(v_1^*, \cdots, v_n^*)$ that satisfy the system (1).

Consider now the function $\sum_1^n v_j^2$ for points satisfying the general linear equation

$$-p + \sum_{1}^{n} \lambda_j v_j = 0 \tag{3}$$

where $\sum_1^n \lambda_j^2 = 1$ ("normal" or "perpendicular" form of the linear equation).

One point satisfying this equation is $(p\lambda_1, \cdots, p\lambda_n)$ for which $\sum v^2$ is obviously equal to p^2. The motivation for choosing this point is based on the 3-dimensional geometrical model. All other points that satisfy (3) have coordinates $(p\lambda_1+\delta v_1, \cdots, p\lambda_n+\delta v_n)$ such that

$$-p + \sum_{1}^{n} \lambda_j(p\lambda_j + \delta v_j) = 0,$$

the δv_j being arbitrary increments. Hence $\sum_1^n \lambda_j \delta v_j = 0$, and for these points

$$\sum_{1}^{n} v_j^2 = \sum_{1}^{n} p^2\lambda_j^2 + 2p \sum_{1}^{n} \lambda_j \delta v_j + \sum_{1}^{n} (\delta v_j)^2 = p^2 + \sum_{1}^{n} (\delta v_j)^2,$$

showing that the function $\sum_1^n v_j^2$ for points on (3) is a minimum for the point $(p\lambda_1, \cdots, p\lambda_n)$, and its value there is p^2.

For equation (2), rewritten in the general form (3),

$$\lambda_j = Q^{-1/2} \sum_{i=1}^{m} k_i a_{ij}, \qquad p = Q^{-1/2} \sum_{r=1}^{m} k_r a_{r0}, \tag{4}$$

where

$$Q = \sum_{s=1}^{n} \left(\sum_{r=1}^{m} k_r a_{rs} \right)^2.$$

The coordinates of its critical point are, therefore,

$$v_j = p\lambda_j = \sum_{i=1}^{m} P k_i a_{ij} = \sum_{i=1}^{m} k_i^* a_{ij}, \qquad j = 1, \cdots, n, \tag{5}$$

on putting $k_i^* = Pk_i$, where P is the constant

$$P = Q^{-1} \sum_{r=1}^{m} k_r a_{r0}. \tag{6}$$

If this point belongs to the set P^* of points that satisfy (1), then it will be

the required minimum solution since the P^* are a subset of the points of (2) and the point $(p\lambda_j)$ marks the minimum of $\sum v^2$ for all points of (2). The set of parameters k_i^* which make the critical point a P^* are found by substituting (5) in (1), resulting in the usual m normal equations, linear in the m variables k_i^*:

$$\left(\sum_{j=1}^{n} a_{ij}a_{1j}\right)k_1^* + \left(\sum_{j=1}^{n} a_{ij}a_{2j}\right)k_2^* + \cdots + \left(\sum_{j=1}^{n} a_{ij}a_{mj}\right)k_m^* - a_{i0} = 0, \qquad i = 1, \cdots m, \tag{7}$$

the solution for which, when substituted back in (5), gives the required least squares solution of the system (1).

With this set of k_i^* it follows from (4) and (6) that the minimum $\sum_1^n v_j^2$ is

$$p^2 = Q^{-1}\left(\sum_{r=1}^{m} k_r a_{r0}\right)^2 = P\sum_{r=1}^{m} k_r a_{r0} = \sum_{r=1}^{m} k_r^* a_{r0}.$$

An n-dimensional geometrical interpretation of this result follows by analogy with the 3-dimensional case for which (1) consists of two linear equations in three unknowns. In the general case the system (1) may be thought of as representing m planes in an n-dimensional Euclidean space with Cartesian coordinates v_j. Equation (2) is any plane containing the intersection of (1). In a Euclidean space $\sum_1^n v_j^2$ is, by definition, the square of the distance from a point in this plane to the origin, and this distance is therefore a minimum for the point (5), an extension to n dimensions of the footpoint of the normal to the plane from the origin. Furthermore, this minimal distance is, by (4), equal to the constant term of the normalized (*i.e.* $\sum_1^n \lambda_j^2 = 1$) linear equation as in the 2 and 3-dimensional cases. The parameters k_i effect a rotation of plane (2) and a corresponding displacement of the footpoint of the normal. When the k_i^* satisfy the normal equations (7) the position of (2) is such that the footpoint lies on the intersection of the planes (1).

THE LINEAR FUNCTIONAL EQUATION

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In many undergraduate courses such as advanced calculus, it is proved that the only solutions of the functional equation

$$f(x + y) = f(x) + f(y) \tag{1}$$

which are continuous, or continuous at a point, are the functions $f(x) = mx$, m a constant. To me, the usual proof, of showing that a solution is linear for x rational and then extending by continuity of the irrationals, lacks elegance, and repeats much of what was done—or should have been done—in setting up the real number system. There are more general hypotheses that imply the same conclusion, for example that f is bounded on some interval, or bounded on a set

of positive measure, or measurable. In this note I prove the first of these generalizations by an argument suitable for undergraduates and which is perhaps shorter than the usual one for the continuous case. The proof is at least not well-known, though I can hardly believe it has escaped notice. Green and Gustin have given an excellent discussion of the history of this equation in [1].

THEOREM. *If $F(x)$ is a solution to* (1) *which is bounded over an interval $[a, b]$, then it is of the form $F(x)=mx$ for some real number m.*

Proof. We show first that $F(x)$ is bounded over $[0, b-a]$. Suppose that for all y in $[a, b]$, $|F(y)|<M$. If x is in $[0, b-a]$, then $F(x+a)$ is in $[a, b]$, so that from

$$F(x) = F(x + a) - F(a),$$

we get

$$|F(x)| < M + F(a).$$

Accordingly, if $b-a=c$, $F(x)$ is bounded in $[0, c]$. Let $m=f(c)/c$, and let $\phi(x)=F(x)-mx$. Then $\phi(x)$ also satisfies (1). We have $\phi(c)=F(c)-mc=0$. It follows that $\phi(x)$ is periodic, with period c, for

$$\phi(x + c) = \phi(x) + \phi(c) = \phi(x).$$

Further, as the difference of two functions bounded over $[0, c]$, $\phi(x)$ is bounded over $[0, c]$, and from the periodicity, $\phi(x)$ is therefore bounded over the entire x-axis.

Suppose x_0 is a number such that $\phi(x_0)\neq 0$. By an easy induction, we have $\phi(nx_0)=n\phi(x_0)$. We can make $|n\phi(x_0)|$ as large as we please by increasing n, which would contradict the boundedness of $\phi(x)$. Therefore $\phi(x)\equiv 0$, or $F(x)\equiv mx$.

The existence of pathological solutions is something in the reach of quite superior students in advanced calculus, and I have several times had students who were able to report (to me) on Jones' example [2] of a pathological solution to (1) that has a connected graph.

It is easy to modify the argument to prove that the only solutions of

$$f(xy) = f(x) + f(y)$$

bounded on an interval $[1, a]$ are of the form $f(x)=k \log x$. Let $\phi(x)=f(x)-f(a)(\log x)/(\log a)$. Instead of periodicity, show that $\phi(ax)=\phi(x)$, which implies that $\phi(x)$ has the same bound in each interval $[1, a]$, $[a, a^2]$, $\cdots$. But $\phi(x^n)=n\phi(x)$.

References

1. J. W. Green and W. Gustin, Quasi-convex sets, Canad. J. Math., vol. 2, 1950, pp. 489–507.
2. F. B. Jones, Connected and disconnected plane sets and the functional equation $f(x)+f(y)=f(x+y)$, Bull. Amer. Math. Soc., vol. 48, 1942, pp. 115–120.

MULTIPLICATION AND DIVISION BY BINOMIAL FACTORS

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Professor Miller in his note *A Pascal triangle for the coefficients of a polynomial*, this MONTHLY, vol. 64, 1957, pp. 268–9, gives in effect an algorithm for obtaining the product of binomial factors. The example he gives is essentially the expansion of $(x-2)(x-3)(x-5)(x-7)$. The process includes the case for $(x+1)^n$, expansion of which gives the binomial coefficients, which may be tabulated in the more usual form:

1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.

TABLE 1

often referred to as Pascal's Triangle, although Pascal himself seems to have given it in the form:

1	1	1	1	1	. . .
1	2	3	4	. . .	
1	3	6	. . .		
1	4	. . .			
1	. . .				

TABLE 2

(see *Oeuvres Complètes de Blaise Pascal*, Paris, 1889, vol. 3, p. 244, fig. 52).

The law of formation of Table 1 is that each term is the sum of the term diagonally above to the left and the term immediately above it; *e.g.*, $6=3+3$. We may also put this as:— $6=1.3+1.3$, where the factors 1 and 1 arise from the fact that the coefficients of $(x+1)$ are 1 and 1. If we had to expand $(x+2)^n$, each term of the appropriate table would be equal to the sum of *twice* the term diagonally above it to the left and *once* the term immediately above it:

1						
1	2					
1	4	4				
1	6	12	8			
1	8	24	32	16		
1	10	40	80	80	32	
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.

TABLE 3

which may be expressed as:

1						
1	2.1					
1	2.2	$2^2.1$				
1	2.3	$2^2.3$	$2^3.1$			
1	2.4	$2^2.6$	$2^3.4$	$2^4.1$		
1	2.5	$2^2.10$	$2^3.10$	$2^4.5$	2^5	
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.

TABLE 4

We may here generalize for the expansion of $(ax+b)^n$:

a				
a	ab			
a	$ab(a+1)$	ab^2		
a	$ab(a^2+a+1)$	$ab^2(2a+1)$	ab^3	
a	$ab(a^3+a^2+a+1)$	$ab^2(3a^2+2a+1)$	$ab^3(3a+1)$	ab^4
a	$ab(a^4+a^3+a^2+a+1)$	$ab^2(4a^3+3a^2+2a+1)$	$ab^3(6a^2+3a+1)$	$ab^4(4a+1)$
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

TABLE 5

Here every term is the sum of b times the term diagonally above it to the left and a times the term immediately above it.

Applying this schema to Professor Miller's example, we have:

$-3+1$	1	-2			
$-5+1$	1	-5	$+6$		
$-7+1$	1	-10	$+31$	-30	
	1	-17	$+101$	-247	$+210$

TABLE 6

i.e. $(x-2)(x-3)(x-5)(x-7)=x^4-17x^3+101x^2-247x+210$.

The coefficients of the binomial factors may be taken in any order; *e.g.*,

$-2+1$	1	-5			
$-7+1$	1	-7	$+10$		
$-3+1$	1	-14	$+59$	-70	
	1	-17	$+101$	-247	$+210$

TABLE 7

The method can be applied where polynomials are involved, and where the coefficient of x in the binomial factors is not unity:

Expand $(x^2+4x-1)(2x-7)(x+5)$.

$$\begin{array}{rl|llll} -7+2 & 1 & +4 & -1 & \\ 5+1 & 2 & +1 & -30 & +7 \\ & 2 & +11 & -25 & -143 \quad +35 \end{array}$$

$$= 2x^4 + 11x^3 - 25x^2 - 143x + 35.$$

If it is necessary to place a trinomial to the left of the vertical line, the mental work involved is slightly more onerous: $(x^2-4x+7)(2x^2+3x+4)$.

$$\begin{array}{rl|llll} 4+3+2 & 1 & -4 & +7 & \\ & 2 & -5 & +6 & +5 \quad +28 \end{array}$$

$$= 2x^4 - 5x^3 + 6x^2 + 5x + 28.$$

The reverse process would of course be division by binomial factors. As an example, in the case of, say, $x^4+4x^3+6x^2+4x+1$ to be divided repeatedly by $x+1$, we have:

1	4	6	4	1
1	3	3	1	
1	2	1		
1	1			
1				

TABLE 8

Here each term is found by adding the term diagonally below it to the left to the term immediately below it; *e.g.* $6=3+3=1.3+1.3$, where again the factors 1 and 1 derive from the coefficients of $x+1$. Generally, if we wish to divide a polynomial expression by the binomial $ax+b$, we may write the resultant polynomial beneath the polynomial to be divided, and each term of the dividend will be equal to the sum of b times the term of the quotient diagonally below it to the left and a times the term of the quotient immediately below it.

Divide $6x^4-29x^3+40x^2-7x-12$ by $3x-4$.

$$\begin{array}{rl|llll} & 6 & -29 & +40 & -7 \quad -12 \\ -4+3 & 2 & -7 & +4 & +3 \end{array}$$

$$= 2x^3 - 7x^2 + 4x + 3.$$

Thus we finally arrive at the method for turning a polynomial into factorial powers, which we define as:

$$x^{(n)} = x(x-1)\cdots(x-n+1)$$

This involves division by $(x-1)$, $(x-2)$, $\cdots$ and so on, and we may set down the work briefly as below:

Find the sum of $x^4-3x^3+5x^2+x-3$.

$$\begin{array}{r|llll} & 1 & -3 & +5 & +1 \quad (-3 \\ -1+1 & 1 & -2 & +3 & (+4 \\ -2+1 & 1 & +0 & (+3 & \\ -3+1 & 1 & (+3 & & \end{array}$$

Whence

$$\Sigma = \frac{(x+1)^{(5)}}{5} + \frac{3(x+1)^{(4)}}{4} + \frac{3(x+1)^{(3)}}{3} + \frac{4(x+1)^{(2)}}{2} - 3x$$

$$= \frac{x}{20}(4x^4 - 5x^3 + 10x^2 + 45x - 34), \text{ on reduction.}$$

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1296. *Proposed by Beckham Martin, Owens-Illinois Glass Co., Toledo, Ohio*

Problem E 1242 [1957, 433] says that the circle orthogonal to the circles (A'), (B'), (C') inscribed in the squares constructed exteriorly (or interiorly) on the sides of a triangle ABC is concentric with the nine point circle of triangle $A'B'C'$. Show, more generally, that all circles cutting the circles (A'), (B'), (C') under angles α, β, γ, respectively, where

$$\cos\alpha \sin A = \cos\beta \sin B = \cos\gamma \sin C,$$

are concentric with the nine point circle of triangle $A'B'C'$.

E 1297. *Proposed by D. C. B. Marsh, Colorado School of Mines*

In Solution II of Problem E 1243 [1957, 434] occurs the following conjecture for proof or disproof: When an integer and its reversal are unequal, their product is never a square except when both are squares.

Show that for any $n>2$ there is a nonsymmetric, nonsquare, n digit integer whose product with its reversal is a square.

E 1298. *Proposed by H. D. Grossman, New York, N. Y.*

It is not difficult to show that the longest linear section of a triangle is the longest side of the triangle. Is the greatest planar section of a tetrahedron the largest face of the tetrahedron?

E 1299. *Proposed by P. L. Chessin, Westinghouse Electric Corporation*

Find the limit of $1+x(1-x[1+x(1-x[1+\cdots])])$, for $|x|<1$.

E 1300. *Proposed by D. A. Robinson, University of Wisconsin*

Let $D_1, D_2, D_3, \cdots$ be the determinants

$$|1|, \quad \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix}, \quad \begin{vmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 3 \end{vmatrix}, \cdots,$$

respectively. Find the value of D_n for any positive integer n.

SOLUTIONS

A Diophantine System

E 1266 [1957, 365]. *Proposed by D. C. B. Marsh, Colorado School of Mines*

Solve $a^3-b^3-c^3=3abc$, $a^2=2(b+c)$ simultaneously in positive integers.

I. *Solution by E. W. Marchand, Eastman Kodak Company.* The first equation may be written $(a-b-c)[a^2+(b-c)^2+ab+bc+ca]=0$. Since the second factor cannot vanish for positive a, b, c, we must have $a=b+c=a^2/2$, whence the only solution in positive integers is $a=2$, $b=c=1$.

II. *Solution by Raymond Huck, Marietta College.* It is seen from the cubic relation that $c<a$, $b<a$. It follows from the quadratic relation that $2(b+c)=a^2<4a$, that $a<4$, and that a is even. Hence the only solution is $a=2, b=c=1$.

III. *Solution by H. M. Gehman, University of Buffalo.* Let us first solve without imposing the restriction that a, b, c be positive integers, but only requiring them to be real. If the second equation is solved for c, and the result substituted in the first equation, a small amount of manipulation gives

$$a(a-2)[(a^2+4a)^2+3(a^2-4b)^2]=0.$$

Hence either $a=0$ and $b=-c$, or $a=2$ and $b+c=2$, or $a=-4$ and $b=c=4$. The only solution in positive integers is $a=2$, $b=c=1$.

Also solved by Leon Bankoff, Merrill Barnebey, Robert Bart, D. M. Bloom, Margaret R. Blue, W. J. Blundon, D. A. Breault, D. M. Brown, P. L. Chessin, Eleanor G. Dawley, Brothers Day, Monte Dernham, Warren Dickinson, N. Ersec, Hazel E. Evans, J. F. Faley, N. J. Fine,

G. R. Flowers, Calvin Foreman, L. D. Goldberg, Michael Goldberg, F. L. Griffin, C. A. Grimm, A. B. Harper, Jr., A. R. Hyde, Edgar Karst, A. F. Kaupe, Jr., M. S. Klamkin, Frank Kocher, Sam Kravitz, Sidney Kravitz, M. A. Laframboise, J. T. Liddle, Joe Lipman, Bernhard Marzetta, J. B. Muskat, Herbert Nadler, T. D. Nagle, Bart Park, Paul Payette, J. L. Pietenpol, C. F. Pinzka, Montfort Plebstnoch, L. A. Ringenberg, Jeff Ritterman, B. S. Sackman, R. A. Sebastian, R. R. Seeber, Jr., R. E. Shafer, Nancy Siljander, Conrado Silva, Arnold Singer, E. P. Starke, R. F. Steinhart, R. S. Underwood, Chih-yi Wang, C. H. Wells, Jr., R. H. Wilson, Jr., the proposer, and one anonymous contributor. Late solution by Seymour Kass.

The proposer remarked that the system $a^3-b^3-c^3=kabc$, $a^2=m(b+c)$, where k, m are positive integers with $k>3m/2$, has no simultaneous solution in positive integers a, b, c. If $k=3$ and $m\geqq 2$, then $a=m$ and b and c are any two positive integers whose sum is m.

A Divergent Series

E 1267 [1957, 365]. *Proposed by Ivan Niven, University of Oregon*

The divergence of the harmonic series $\sum 1/n$ is often established by comparison with the obviously divergent series $\sum f(n)$ where $f(n)=2^{-k}$, the integer k being defined by the inequality $2^k\geqq n>2^{k-1}$. Establish the convergence or divergence of the series $\sum(1/n-f(n))$.

Solution by C. F. Pinzka, Xavier University, Cincinnati, Ohio. Since $\sum_{n=1}^{2^r} f(n)=r/2+1$ and $\sum_{n=1}^{2^r} 1/n>\int_1^{2^{r+1}} dx/x>r\ln 2$, $\sum_{n=1}^{2^r}(1/n-f(n))>r(\ln 2-1/2)-1$ and the series diverges.

Also solved by Julian Braun, T. S. Chihara, W. H. Colbert, Jr., N. J. Fine, D. S. Greenstein, F. L. Griffin, R. H. Hou, A. R. Hyde, N. D. Kazarinoff, J. F. Kennison, M. S. Klamkin, Joe Lipman, D. C. B. Marsh, Bernhard Marzetta, J. B. Muskat, Paul Payette, L. A. Ringenberg, Jeff Ritterman, Michael Rosen, R. E. Shafer, G. B. Thomas, Jr., and the proposer.

A Special Circulant

E 1268 [1957, 365]. *Proposed by A. J. Goldman and O. S. Wolf, Princeton University*

Evaluate the determinant D_n which has $(1, \cdots, n)$ as first row, $(2, \cdots, n, 1)$ as second row, *etc.*

Solution by A. E. Danese, Eastman Kodak Company. It is just as easy to evaluate the determinant which has $(a, a+d, \cdots, a+(n-1)d)$ as first row, $(a+d, a+2d, \cdots, a)$ as second row, *etc.* Take $n>2$. By adding all subsequent rows to the first row, we obtain $na+n(n-1)d/2$ as a factor. Removing this and performing the column operations c_r-c_n, $r=1, \cdots, n-1$, we obtain d^{n-1} as a factor. Now we can readily reduce the order of the determinant by one. In the reduced determinant we perform the column operations $c_r+(n-r)c_1$, $r=2, \cdots, n-1$, to obtain a determinant having only zeros below the secondary diagonal, only ones in the first column, and all remaining elements equal to n. This is readily evaluated so as to yield for the final result

$$(-1)^{n(n-1)/2}(nd)^{n-1}[a+(n-1)d/2],$$

which proves to be true also for $n=1$ or 2, and which for $a=d=1$ becomes

$$(-1)^{n(n-1)/2}\, n^{n-1}\,(n+1)/2.$$

These results occur in Muir, *A Treatise on the Theory of Determinants*, New York, 1933, pp. 442–444.

Also solved by Robert Bart, A. P. Boblétt, Julian Braun, K. A. Bush, F. M. Djoup, Jr., and A. G. Konheim (jointly), A. L. Epstein, N. Ersec, N. J. Fine, G. R. Flowers, J. F. Foley, Calvin Foreman, H. M. Gehman, Lee Goldberg, G. A. Harris, Jr., R. H. Hou, A. R. Hyde, P. G. Kirmser, M. S. Klamkin, M. A. Laframboise, Joe Lipman, R. L. London, Marshall Luban, E. W. Marchand, D. C. B. Marsh, Bernhard Marzetta, W. B. Morgan, W. L. Murdock, J. B. Muskat, C. O. Oakley, T. J. Pignani, C. F. Pinzka, E. J. F. Primrose, L. A. Ringenberg, Jeff Ritterman, D. A. Robinson, R. E. Shafer, D. D. Strebe, Ling-Erl Ting, Chih-yi Wang, Alan Wayne, David Zeitlin, and the proposers.

Editorial Note. The generalized problem has also been located in G. Kowalewski, *Determinantentheorie*, Chelsea, 1948, p. 109. Both the proposed problem and the generalization are solved in Problem 2774 [1920, 235].

A Roulette Problem

E 1269 [1957, 365]. *Proposed by Frank Kocher, Pennsylvania State University*

Prove that the area under one arch of the curve generated by a vertex of a regular polygon rolling on a straight line is equal to the area of the polygon plus twice the area of its circumscribed circle.

Solution by Michael Goldberg, Washington, D. C. The area under the arch can be divided by straight lines from the cusps to the stationary points into $n-1$ sectors and $n-2$ triangles. The lengths r_i of the straight lines are sides or diagonals of the polygon. Hence the triangles correspond to the $n-2$ triangles into which the polygon is cut by diagonals from a vertex.

The area of each sector is $\pi r_i^2/n$, where r_i is the length of a radius. The sum of the areas of the sectors is $(\pi/n)\Sigma r_i^2$. If unit masses are placed at the vertices of the polygon whose circumcircle is of radius R, then the total moment of inertia of these masses about the center is nR^2. Therefore the total moment of inertia about a vertex is nR^2+nR^2, since the center of gravity is translated the distance R. But this is the same as Σr_i^2. Hence $(\pi/n)\Sigma r_i^2=(\pi/n)2nR^2=2\pi R^2$.

When n becomes infinite, the familiar area of $3\pi R^2$ under the cycloid is obtained.

Also solved by Robert Bart, R. C. Brown, W. B. Carver, N. J. Fine, A. R. Hyde, L. M. Lewandowski, Joe Lipman, D. C. B. Marsh, Paul Payette, L. A. Ringenberg, Jeff Ritterman, F. G. Schmitt, Jr., Chih-yi Wang, and the proposer.

Regular Simplexes

E 1270 [1957, 365]. *Proposed by Leo Moser, University of Alberta*

What is the smallest positive even integer n such that in both n and $n+1$ dimensions the regular simplex of edge 1 will have a rational number as its content? (Dedicated to Professor H. S. M. Coxeter.)

Solution by W. J. Blundon, Memorial University of Newfoundland. The content of the regular simplex of edge 1 in E_n is given by $\{(n+1)/2^n\}^{1/2}/n!$. Since n is to be even, we must have $n+1=x^2$ and $(n+2)/2=y^2$, where x and y are positive integers. This yields the Pellian equation $x^2-2y^2=-1$, successive solu-

tions of which are (1, 1), (7, 5), (41, 29), (239, 169), · · · . Since $x>1$, min $x=7$, so that min $n=48$. The next smallest values of n are clearly $41^2-1=1680$ and $239^2-1=57120$.

Also solved by N. J. Fine, Calvin Foreman, Michael Goldberg, N. W. Johnson, Paul Payette, and the proposer.

Editorial Note. For the content of a regular simplex see D. M. Y. Sommerville, *An Introduction to the Geometry of n Dimensions*, p. 126, or H. S. M. Coxeter, *Regular Polytopes*, p. 295.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4768. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let O be the center of the circumcircle (O) of a regular convex polygon $A_1A_2A_3 \cdots A_{25}A_{26}$ of 26 sides, and let O_1 and O_2 be symmetric to O with respect to the lines $A_{25}A_1$ and A_2A_6. Prove that O_1O_2 is equal to a side of an equilateral triangle inscribed in (O).

4769. *Proposed by H. Schwerdtfeger, University of Melbourne, Australia*

(1) Find all analytic functions $f(z)=u+iv$ of the complex variable $z=x+iy$, whose real part appears in the form $u=g(x)+h(y)$, where $g(x)$ and $h(y)$ are two real continuous functions of the real variables x and y whose first and second derivatives are continuous.

(2) The same except that $u=g(x)\cdot h(y)$.

4770. *Proposed by J. R. Holdsworth and J. R. Smart, China Lake, California*

Determine whether the following series is divergent or not:

$$S=\sum_{n=1}^{\infty}\sin^n n.$$

4771. *Proposed by Leonard Carlitz, Duke University*

Muir (*Contributions to the History of Determinants* 1900–1920, London and Glasgow) reproduces the assertion of Vogt that the determinant

$$D_n = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ 2 & 2^2 & \cdots & 2^n & 2^{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ n & n^2 & \cdots & n^n & n^{n+1} \\ \frac{n}{2} & \frac{n^2}{3} & \cdots & \frac{n^n}{n+1} & \frac{n^{n+1}}{n+2} \end{vmatrix}$$

vanishes for $n \geqq 2$ and even. Prove the truth of this assertion. Also show that $D_n \neq 0$ for n odd.

4772. *Proposed by M. S. Klamkin, AVCO Research and Development, Lawrence, Mass.*

It is easy to show that there exist consecutive prime pairs such that their difference is arbitrarily large. Do there exist consecutive prime triplets P_1, P_2, P_3 such that min (P_2-P_1, P_3-P_2) is arbitrarily large?

SOLUTIONS

Minimum Least Common Multiple

3834 [1937, 394]. *Proposed by Paul Erdös.*

Let $a_1<a_2< \cdots <a_n \leqq 2n$ be a sequence of positive integers. Then

$$\min [a_i, a_j] < 6([n/2] + 1),$$

where $[a_i, a_j]$ denotes the least common multiple of a_i and a_j. This is the best possible estimate.

Solution by C. R. Phelps, Rutgers University. The stated conclusion is invalid; for, when $n=3$, the sequence 3, 4, 5 has min $[a_i, a_j]=12=6([n/2]+1)$, and when $n=4$, the sequence 5, 6, 7, 8 has min $[a_i, a_j]=24>6([n/2]+1)$. However, the following corrected conclusion is valid:

$$\min [a_i, a_j] \leqq 6([n/2] + 1), \qquad n \neq 4.$$

Consider first, for any n, the special sequence in which $a_i=n+i$. The first even number in this sequence is $2([n/2]+1)$, and the number $3([n/2]+1)$ is in the sequence also, for $n=3$ and $n \geqq 5$. Hence min $[a_i, a_j] \leqq 6([n/2]+1)$ for this special sequence, except when $n=4$, as the conclusion is evident for $n=2$.

For any other sequence $a_1<a_2< \cdots <a_n \leqq 2n$, for any a_i there exists an integer $k_i \geqq 1$ such that $n<k_ia_i \leqq 2n$. If $k_ia_i=k_ja_j$ for some i, j, then this $[a_i, a_j]$ divides k_ia_i, and is thus $\leqq 2n$. Otherwise the k_ia_i, $i=1, \cdots, n$, are all distinct and exactly the special sequence in some order, so that for some p, q, k_pa_p and k_qa_q are precisely $2([n/2]+1)$ and $3([n/2]+1)$, whence $[a_p, a_q]$ divides $6([n/2]+1)$. Thus, in any case, min $[a_i, a_j] \leqq 6([n/2]+1)$.

We show also that $6([n/2]+1)$ is the exact minimum for the special sequence, for each n, and hence the best possible value. Suppose, in the special sequence, that $[a_i, a_j]=M<6([n/2]+1)$, with $a_i<a_j$. Then $a_j \geqq n+3$, so that $a_j>2([n/2]+1)$, with $M=ma_j$ for some integer m. Hence $m=2$, and $M=ka_i=2a_j$, so that $k \geqq 3$. If $k=3$, then $a_i<2([n/2]+1)$ with a_i even, an impossibility. If $k \geqq 4$, then $a_i<(3/2)([n/2]+1)$, so that $a_i<n+1$, also impossible. Hence $M<6([n/2]+1)$ is impossible.

Disjoint Permutations

4647 [1955, 447; 1957, 277]. *Proposed by James Munkres, Los Alamos Scientific Laboratory*

A permutation of the integers $1, \cdots, n$ is called an n-chain; two n-chains are disjoint if any two integers which are adjacent in one chain are not adjacent in the other. (The first integer is considered adjacent to the last for this purpose.) Does there exist, for each n, a collection containing $[(n-1)/2]$ mutually disjoint n-chains?

Editorial Note. III. Both Roberto Frucht and Haim Hanani have pointed out that the problem is classical (as your Editor should have remembered) being given by Lucas, (*Récréations Mathématiques*, Paris, 1891, t. II, pp. 162–168) under the title, *Les Rondes Enfantines*. The same discussion is given in Kraitchik, *Mathematical Recreations*, New York, 1942, pp. 227–229.

Lucas' solution may be summarized as follows. Write down the number n and, to its right, the numbers 1 and 2. Then put 3 in the 5th place, 4 in the 7th place, *etc.* When the last odd place has been filled put the remaining numbers (in order) in the even places returning from right to left. For example with $n=17$:

17, 1, 2, 16, 3, 15, 4, 14, 5, 13, 6, 12, 7, 11, 8, 10, 9.

The $[(n-1)/2]$ other required permutations are obtained, each from the preceding, by holding n fixed, replacing $(n-1)$ by 1, and increasing each of the other numbers by unity. For the case above we continue:

17, 2, 3, 1, 4, 16, 5, 15, 6, 14, 7, 13, 8, 12, 9, 11, 10;
17, 3, 4, 2, 5, 1, 6, 16, 7, 15, 8, 14, 9, 13, 10, 12, 11;

and so on until eight lines are complete.

That this procedure meets the requirements is almost obvious from the simple pattern. Lucas gives a geometrical device from which the proof is evident upon noticing that, because of the slopes of the lines, no two points can be joined by more than one line in all the $[(n-1)/2]$ designated paths.

Also solved by J. J. Harkema.

Binomial Coefficients

4723 [1957, 116]. *Proposed by Joachim Lambek, McGill University*

Given a nonnegative integer n and a prime p, obtain an expression for the number of binomial coefficients $\binom{n}{r}$ which are not divisible by p. (This generalizes problem I, 7 of the Putnam Competition Examination for 1956. See this MONTHLY, vol. 64, 1957, p. 24.)

Editorial Note. Write n in the scale of p, and let n_i be the digits which appear. Then the re-

quired number is $\prod(n_i+1)$. This is Theorem 2 of the note by N. J. Fine, *Binomial coefficients modulo a prime*, this MONTHLY, vol. 54, 1947, pp. 589–592.

Also solved by Robert Breusch, L. Carlitz, A. L. Davis, Harley Flanders, Emil Grosswald, A. P. Hillman and C. T. Long and H. Schneider, J. H. Hodges, Joe Lipman, D. C. B. Marsh, J. B. Roberts, Leopold Sauve, L. K. Williams, Chih-yi Wang, and the proposer. Late solution by F. W. Perkins.

Normal Matrices

4725 [1957, 116]. *Proposed by Olga Taussky, National Bureau of Standards*

A square $n\times n$ matrix $A=(a_{ik})$ with complex numbers as elements is called normal if $AA^*=A^*A$, where A^* is the transposed and complex conjugate of A. A matrix A is called nilpotent if for some integer $r\geqq 1$ the matrix A^r is the zero matrix 0. Prove (by rational methods only) that a normal and nilpotent matrix is the zero matrix.

I. *Solution by P. P. Saworotnow, The Catholic University of America.* It is easy to see that $A=0$ if and only if $AA^*=0$. Indeed, let $AA^*=(b_{ij})$, then $b_{ii}=\sum_{k=1}^{n}|a_{ik}|^2$ and so $b_{ii}=0$ only if each $a_{ik}=0$, $k=1,\cdots,n$.

It follows that if $A\neq 0$ and $A=A^*$ then $A^2\neq 0$ and $A^{2^m}\neq 0$ for each integer m and hence $A^r\neq 0$ for every integer r.

Now let A be normal and suppose $A^r=0$, then $(AA^*)^r=A^r(A^*)^r=0$. But then $AA^*=0$ which in turn implies $A=0$.

II. *Solution by Howard Eves, University of Maine.* A proof may be based on the well known fact that a complex normal matrix is unitarily similar to a diagonal matrix. Thus there is a nonsingular matrix P and a diagonal matrix D such that $D=P^{-1}AP$. Then $D^r=P^{-1}A^rP=0$, and, since D is diagonal, $D=0$. Therefore $A=PDP^{-1}=0$.

Also solved by Kurt Bing, Robert Breusch, T. A. Brown, L. Carlitz, P. M. Cohn, Harley Flanders, D. S. Greenstein, Ronald Jacobowitz, D. S. Kahn, R. L. McFarland, Albert Madansky, D. C. B. Marsh, M. H. Pearl, R. J. Pegis, H. Schneider, Melvin Schwartz, H. Schwerdtfeger, D. P. Squier, Hirofumi Uzawa, Chih-yi Wang, and the proposer.

Concurrent Lines and a Conic

4726 [1957, 117]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If the parallels to the asymptotes of a conic (C), drawn through an arbitrary point P of its plane, intersect (C) in P_1 and P_2, if the perpendiculars to PP_1 and PP_2 at P_1 and P_2 intersect in a point O, and if the polar of P with respect to (C) intersects the conic in M_1 and M_2, then the perpendicular bisector of segment M_1M_2 passes through O.

Solution by Joe Lipman, University of Toronto. Let the lines PP_1 and PP_2 intersect the line M_1M_2 in K_1 and K_2, and intersect the ideal line in I_1 and I_2 respectively. Then the points P, P_1, K_1, I_1 are a harmonic range, whence P_1 is the midpoint of PK_1. Similarly P_2 is the midpoint of PK_2. Therefore O is the circumcenter of triangle PK_1K_2 and the perpendicular from O upon K_1K_2 bisects it. It remains only to show that the midpoint of K_1K_2 coincides with the midpoint of M_1M_2.

Let P be $(0, 0)$, and let

$$(C) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \tag{1}$$

$$M_1M_2 \equiv gx + fy + c = 0, \tag{2}$$

$$(PP_1, PP_2) \equiv ax^2 + 2hxy + by^2 = 0. \tag{3}$$

Now the coordinates of the midpoint of M_1M_2 are found by taking $-\frac{1}{2}$ times the ratio of the coefficient of the linear term to that of the quadratic term in the equation resulting when y and x, respectively, are eliminated between (2) and (1), or, equivalently, between (2) and

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c - 2(gx + fy + c) = ax^2 + 2hxy + by^2 - c = 0. \tag{4}$$

The midpoint of K_1K_2 is found by a similar treatment of (2) and (3). Since (3) and (4) differ only in the absolute term, the results of both treatments will be the same.

Also solved by W. B. Carver, R. Deaux, Josef Langr, O. J. Ramler, P. Somanadham, Chih-y Wang, and the proposer.

Divergent Integral and Convergent Series

4727 [1957, 117]. *Proposed by G. R. MacLane, the Rice Institute*

Find a function $f(x)$, upper semi-continuous and nonnegative on $[0, \infty)$, bounded on each finite interval $(0, T)$, such that $\int_0^\infty f(x)dx = \infty$ and $\sum_{n=1}^{\infty} f(nh) < \infty$ for every $h>0$. (*Cf.* problem 4670, [1957, 119].)

Solution by the Proposer. We construct a sequence of closed nowhere dense sets E_n, $E_n \subset (n, n+1)$, $n \geq 1$, such that $m(E_n)>0$ and such that: if $rh \in \bigcup_1^n E_k$, $sh \in E_{n+1}$ for some $h>0$ and integers r and s, then $h<1/(n+1)$. From such a sequence an f of the required sort is easily constructed:

$$f(x) = \begin{cases} 0, & x \notin \bigcup_1^\infty E_n \\ 1/m(E_n), & x \in E_n,\ n \geq 1. \end{cases}$$

Set $aE_n = \{x \mid x/a \in E_n\}$. The sequence E_n is chosen inductively. Let E_1 be any closed nowhere dense set of positive measure in $(1, 2)$. Assuming that $E_1, \cdots, E_n$ have been determined, set

$$F_n = \bigcup_{r,s} \left(\bigcup_{k=1}^{n} \frac{s}{r} E_k \right),$$

where $\bigcup_{r,s}$ is the union for all rationals s/r with $r \leq (n+1)^2$. Now F_n is clearly nowhere dense; hence $(n+1, n+2)$ contains an open interval disjoint from F_n. Choose E_{n+1} to be any closed *etc.*, subset of this interval. The required property follows, for if $rh \in \bigcup_1^n E_k$ and $h \geq 1/(n+1)$, then $n+1 \geq rh \geq r/(n+1)$, or $r \leq (n+1)^2$. Thus $sh \in \bigcup_1^n (s/r)E_k \subset F_n$, and hence $sh \notin E_{n+1} \subset F_n'$.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, vol. I: Theory of Statistics. Ed. by Jerzy Neyman. University of California Press, Berkeley and Los Angeles, 1956. ix+208 pp. $6.00.*

This volume, along with its four companion volumes, joins the *Proceedings* of the two earlier Berkeley symposia as a source of fascinating results, stimulating conjectures, and novel formulations. Warm thanks should be expressed to Jerzy Neyman, organizer and editor of the symposium, and to his associates, for their efforts in arranging and publishing these *Proceedings*.

Sixteen papers appear in this book, representing the work of seventeen authors. The following outline crudely classifies these papers and very briefly describes their contents:†

Nonparametric analysis

Z. W. Birnbaum. Upper confidence limit for Pr $\{Y<X\}$.

B. L. van der Waerden. Two-sample test based on ranks transformed by the inverse normal cumulative function‡.

J. L. Hodges, Jr. and E. L. Lehmann. Stochastic approximation (Robbins-Monro) methods; choice of the sequence of coefficients to be used.

Aryeh Dvoretsky. Stochastic approximation methods in a very general framework.§

Parametric analysis

Joseph Berkson. Comparisons of first and second moments for six estimators in quantal bio-assay.

L. LeCam. Asymptotic estimation and testing theory, extending the work of Neyman and Wald.

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* By a special arrangement, members of the Mathematical Association of America may purchase any or all of the five volumes of the *Proceedings* at a 25% discount. Orders must be sent to the Statistical Laboratory, University of California, Berkeley 4, California. Reference must be made to membership in the Association and checks, made out to the University of California Press for the correct amount, must accompany orders.

† A more detailed version of this review (including references to relevant material published later) will be sent, on request to the reviewer, until the store of copies is exhausted.

‡ Tables for this test have since appeared. (B. L. van der Waerden and E. Nievergelt, Tafeln zum Vergleich Zweier Stichproben Mittels X-test und Zeichentest, Berlin, 1956.)

§ For a subsequent simplification, see J. Wolfowitz, On stochastic approximation methods, Ann. Math. Statist., vol. 27, 1956, pp. 1151–1156. For an expository article, see Cyrus Derman, Stochastic approximation, Ann. Math. Statist., vol. 27, 1956, pp. 879–886.

Herman Chernoff and Herman Rubin. Estimation of the unknown point of discontinuity of a probability density function having only one such point.

Charles Stein. The usual estimator for the mean of a multivariate normal distribution with three or more components, and with known covariance structure, is inadmissible with respect to the natural quadratic loss function.

Samuel Karlin. Detailed and extensive results on two-decision theory for the general exponential and still more general Pólya type distributions.

Sylvain Ehrenfeld. Complete class theorems for experimental designs in the linear hypothesis context.

G. Elfving. Allocation theory for linear hypothesis designs when repeated observations under the same conditions cannot be made arbitrarily often.

Intermediate between parametric and nonparametric analysis or not readily classified in these terms.

Wassily Hoeffding. Effects on statistical techniques of nonfulfillment of assumptions, *e.g.*, normality. Distinguishability of two sets of distributions.

Charles Stein. A connection between parametric and nonparametric theory relating to the existence of a one-dimensional problem for each state of nature that is asymptotically as difficult as the original problem.

Herbert Robbins. Bayes estimation when information about the prior distribution is accumulated in a sequence of observations.

Ulf Grenander and Murray Rosenblatt. Estimation of the spectral density of a stationary stochastic process.

Murray Rosenblatt. Comparison of Markov and least-squares estimators for estimating the regression parameters of a stochastic process in which a stationary residual process is superimposed on a mean function linear in the unknown regression parameters.*

The physical format of this volume is first-rate in all respects and I noticed very few typographical errors. The book suffers from two probably inevitable difficulties in any symposium publication. First, because of the fixed presentation date, about a quarter of the papers are of a progress-report nature and would probably not have been published until later were it not for the existence of the symposium. (This characteristic has merits as well as drawbacks.) Second, the absence of normal refereeing has resulted in lack of clarity in some of the papers.

WILLIAM KRUSKAL
University of Chicago

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* The approach of Grenander and Rosenblatt to time series is expounded in their recent book, Statistical Analysis of Stationary Time Series, New York, 1957.

Fundamental Mathematics. By Leslie H. Miller. Henry Holt, New York, 1957. 323 pp. $3.50.

This book is a text and a workbook to be used for a remedial course on the freshman level in colleges. I like the introduction of algebraic notations and concepts in the arithmetic portion of the book, because it gives the student a better understanding of the arithmetic and prepares him for the formal algebra part of the book. More emphasis is placed on the comprehension of the fundamental concepts than in the other remedial workbooks with which I am familiar. With Chapters 10, 11, 12 on "Quadratic Equations," "Common Logarithms," and "Preview of Trigonometry," the book can be used for a three- or four-hour course per week for one semester. It can be used to advantage in Teachers Colleges for those going into elementary work.

I believe Professor Miller has taken care of the very heterogeneous group that one finds in a remedial class in college. Most of his material is of a very elementary nature, but there are still enough challenging ideas to keep the better of the poor students from becoming bored. It is very unfortunate that we have students in college who need an elementary remedial course; but since this is the situation, I know of no better book for the review than this workbook by Professor Miller.

JOHN C. KNIPP
University of Pittsburgh

Numerical Methods. By R. A. Buckingham. Pitman, New York, 1957. xii+597 pp. $15.00.

This is the most comprehensive book on computational techniques the reviewer has yet seen. The book is oriented toward the computer with a desk calculator rather than a high-speed electronic calculator, but can be of real value to both. In spite of its rather high price it should be in the library of every computing installation at least, and preferably on the desks of those who are called analysts in the high-speed digital computing industry.

The first chapter is an introduction to computation. The next five chapters are on polynomial fits to functions by the applications of the various differencing techniques. These fits are used for interpolation, integration and differentiation of tabulated functions.

The real worth of this book makes itself felt almost immediately. Not only does the author have comprehensive discussions of practically every technique known, with detailed references to the literature, but also he transmits a kind of "feel" for the subject by noting when each formula is particularly powerful or useful and the various pitfalls a computer is likely to fall into with each formula. In the reviewer's opinion it is this, not strictly mathematical, advice that makes this book outstanding in its field.

Chapters 7 and 8 are concerned with both approximate and recurrence methods of solving ordinary differential equations. Various types of equations and both single and two-point boundary conditions are discussed. Again there are extensive references to the literature and many helpful hints and warnings on each method discussed.

The next five chapters are on various algebraic problems such as finding the zeros of polynomials, least-squares fits, and simultaneous equations. These chapters contain comments on essentially every known useful technique including a good discussion of relaxation methods.

Chapter 14 contains discussions of matric and vector techiques of solving ordinary differential equations and certain integral equations.

Chapter 15 discusses polynomial fits by difference techniques to multivariable data. Many formulas are derived and the techniques explained for deriving other special-purpose formulas.

The last chapter contains a discussion of the partial differential equations readily susceptible to numerical solution and expounds these techniques with some emphasis on relaxation methods. These later chapters too are well referenced and full of helpful hints.

The book closes with four appendices on

(1) Relations Between Powers and Factorials.
(2) Summary of Difference Formulas, with Remainders.
(3) Lagrangian Formulas for Differentiation and Integration.
(4) Orthogonal Polynomials.

There is then an "Index of Useful Formulas in Text," which the reviewer has found very convenient, and finally a fairly complete index.

The book has with each important technique carefully worked out examples with comments keyed into them. At the end of each chapter is a collection of problems.

This book is easy to read and understand. It is very good as a reference work and should have a wide appreciative audience. The book seems a bit formidable as a text for a formal course in numerical analysis, though it could be used with considerable picking and choosing by the instructor.

JOHN E. MAXFIELD
Naval Ordnance Test Station,
China Lake, California

I Am a Mathematician. By Norbert Wiener. Doubleday, Garden City, N. Y., 1956. 380 pp. $5.00.

Professor Norbert Wiener is a pioneer in many fields, In *I Am a Mathematician* he has done it again. In autobiographical form the story begins in 1919 when Wiener came to Massachusetts Institute of Technology as an instructor in the department of mathematics, and contains a chronological account of his work, his travels, his scientific contacts, and his personal experiences during the next thirty-five years.

Although the book is rich in anecdote, the major emphasis in the story is on Wiener's intellectual development and his relation to the forward movement of scientific ideas during the period. He has taken considerable pains to discuss scientific concepts in everyday language and to illustrate their meanings by examples which are within the experience of the average citizen. Thus he gives a readable account of his early work on such abstract topics as Brownian motion, harmonic analysis, vector spaces, and quantum mechanics, relating each to its historical setting and to other fields.

Many factors influenced the development of his career. Most important was the fact that from the first he was convinced of the necessity to the mathematician of maintaining contact with the physical interpretations of purely mathematical problems. In many instances he found that his physical sense determined the images he formed and the tools by which he sought to solve problems. His frequent visits abroad to study, to lecture, and to attend scientific meetings brought him the thrill of meeting young contemporaries and the inspiration of knowing the older mathematicians whom he regarded as "links connecting us with the great past of mathematics," and he gives many vivid sentence descriptions of these people.

Of particular interest in the technical account of his later work is his analysis of the high-speed digital computer, and of the parallelism between computing machines and the human brain and nervous system. He includes also an interesting chapter giving his views on the social responsibility of a scientist. A scholar, he concludes, is not justified in withholding his research. His ideas belong to the times rather than to himself. He should, insofar as he is able, make public the social and economic dangers associated with his inventions.

As the record of his career unfolds, we see how the individual pieces of work he has done have built themselves into an organized pattern, coming inevitably to a climax in Cybernetics.

Although the book is primarily the story of ideas, it is no less the story of the man whose consecration and scientific discipline were such that he was able to produce those ideas. Despite his genius, his success was not achieved without hard work. In his early years at M.I.T., he taught twenty hours a week and still found time to study and to create mathematics. He had to overcome indifference and a certain amount of antipathy born of jealousy and social incompatability. Work in an active field of mathematics is at best a highly competitive business, so he was often working under pressure. He was hampered also by physical and emotional handicaps, and he had to experience the usual difficulties which come to a father of two children. Nevertheless he remained consecrated to the pursuit of truth. Nothing for him was greater than the thrill of creation. The task of the mathematician, he says, is "to use a rigid and demanding medium to express a new and significant vision of some aspect of the universe." He is therefore no less a creative artist than the painter or the musician, even though the strict discipline necessary to appreciate the aesthetic qualities of mathematical work means that only a few are able to recognize a

powerful theory or a clever and elegant proof.

In the opinion of this reviewer, Professor Wiener's book is both meaningful and readable for the educated reading public, and of especial interest to students and teachers of mathematics and other sciences. In particular, it should provide useful collateral reading for a "general education" course in mathematics.

JEANNE AGNEW
Oklahoma State University

BRIEF MENTION

Insights into Modern Mathematics. Twenty-third Yearbook. The National Council of Teachers of Mathematics, Washington, D. C., 1957. viii+440 pp.

The National Council of Teachers of Mathematics should indeed be proud in their achievement. This volume provides a careful treatment of representative topics in modern mathematics. The expositions are clear and have been developed on the assumption that the reader has about two years of college mathematics. Yet there has been no attempt to "water down" the mathematics of a more advanced level. This is indeed a book which every teacher of mathematics, either high school or college, should welcome. It is this reviewer's sincere hope that the effect of this volume, both upon college and high school teaching, will be profound. This does not mean that the subject matter of the twenty-third yearbook should be taught in these classes, but rather that it will help provide the teacher with the background necessary to make intelligent choices. There is considerable new mathematics which has been discovered recently, but the biggest change in high school is in *emphasis*. The reader can gain insight into the contents of the book by noting its thirteen chapters and the competence of the authors thereof:

I. Introduction, by Carroll V. Newsom.
II. The Concept of Number, by Ivan Niven.
III. Operating with Sets, by E. J. McShane.
IV. Deductive Methods in Mathematics, by Carl B. Allendoerfer.
V. Algebra, by Saunders MacLane.
VI. Geometric Vector Analysis and the Concept of Vector Space, by Walter Prenowitz.
VII. Limits, by John F. Randolph.
VIII. Functions, by Rudolph E. Langer.
IX. Origins and Development of Concepts of Geometry, by S. H. Gould.
X. Point Set Topology, by R. H. Bing.
XI. The Theory of Probability, by Herbert Robbins.
XII. Computing Machines and Automatic Decisions, by Charles B. Tompkins.
XIII. Implications for the Mathematics Curriculum, by Bruce E. Meserve.

Congratulations to the National Council of Mathematics on the publication of this, the most eximious of their yearbooks.

Engineering Analysis (*A Survey of Numerical Procedures*). By Stephen H. Crandall. McGraw-Hill, New York, 1957. x+417 pp. $9.50.

It is always a pleasure to welcome another of the Engineering Societies Monographs, which the McGraw-Hill Book Company publishes under the auspices of the American Society of Civil Engineers, American Institute of Mining and Metallurgical Engineers, the American Society of Mechanical Engineers, and the American Institute of Electrical Engineers. Among other problems treated from an engineering viewpoint in this book are the eigenvalue problems for both continuous and discrete systems. The use of finite difference methods and relaxation procedures may seem a bit startling to a few of the old guard, but is certainly compatible with current engineering practice. A mathematician who wishes to keep abreast of engineering students of today, could well spend some profitable hours with this monograph.

Pure Mathematics (*A First Course*). By J. K. Backhouse and S. P. T. Houldsworth. Longmans, Green, New York, 1957. xi+472 pp. $2.60.

It is always interesting to compare our current textbooks with the modern books from England. The current American books seem to be in much more of a state of flux than those of our English cousins, if this book is a typical example.

Basic Mathematics for Radio and Electronics. By F. M. Colebrook and J. W. Head. Philosophical Library, New York, 1957. 359 pp. $6.00.

Mathematics for Science and Engineering. By Philip L. Alger. McGraw-Hill, New York, 1957. xi+360 pp. Text edition, $5.50; Trade edition, $6.95.

This revision of Steinmetz's *Engineering Mathematics* should be an interesting book to teach, particularly for students who need some brush-up between calculus and their modern engineering mathematics. This reviewer recommends it for your consideration.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

NRC COMMITTEE ON APPLICATIONS OF MATHEMATICS

In 1954 the Division of Mathematics of the National Research Council established a standing Committee on Applications of Mathematics with the following functions:

(a) To facilitate cooperation among organizations concerned with various aspects of mathematics in applied settings.
(b) To call attention to the emergence of new areas in which significant applications of mathematics may be possible.
(c) To serve as a focus for the continuing scrutiny of problems concerned with training and research in mathematics as related to its applications.
(d) To take whatever steps are deemed appropriate to enhance the effectiveness of mathematics in its applications.

Dr. A. S. Householder, Oak Ridge National Laboratory, is now Chairman of this Committee. This Committee has been concerned with problems of training at the high school level. While doubtless many factors contribute in tending to repel even superior students from courses in mathematics, one of the important factors is lack of information about the careers that are open to mathematicians, and a scant and often distorted conception of mathematicians and their activities. The Committee felt that a good contribution toward making mathematics interesting might be by attempting to make mathematicians interesting. It seemed that this could be done best by presenting sketches, or profiles in the New Yorker sense, of living mathematicians.

The National Science Foundation has provided funds for interviews and for writing the profiles. It is hoped that eventually these profiles will be printed in a brochure to be made available at no cost to high school students, as well as their teachers and counsellors. It is planned to include in this brochure a statement regarding mathematical requirements of various occupations.

The Chairman of this Committee welcomes suggestions for future projects.

1958 COLLEGE-INDUSTRY CONFERENCE OF ASEE

"New Concepts in the Education and Development of Technical Manpower" is the theme of the 1958 College-Industry Conference of the American Society for Engineering Education to be held at the University of Michigan, January 30–31.

Four speakers are scheduled to present the new concepts arising from current research: Paul Reinert, President of St. Louis University, *New Concepts in Education from the President's Committee on Education Beyond High School;* Andrew A. Kucher, Vice President of Engineering and Research for the Ford Motor Company, *New Concepts in Engineering;* Donald C. Pelz, University of Michigan Survey Research Center, *New Concepts Useful in Motivating Engineering and Technical Personnel;* Lee Danielson or John W. Riegel, U-M Bureau of Industrial Relations, *New Management Concepts Obtained Through Interviews with Employed Engineers and Technicians.*

Four additional speakers will describe new programs designed by their companies to utilize new concepts of professional development: W. R. Collings, President of Dow-Corning Corporation; E. E. Sivacek, Chief Engineer, King-Seeley Corporation; A. L. Bain, Superintendent of Engineering Training for Western Electric Company; L. V. French, Director of Engineering Personnel for Whirlpool-Seeger Corporation. These reports hold many implications for education as well as for employers of engineers.

Registration will be held from 8:30 to 10:00 A.M., January 30 and the conference will adjourn at 2:00 P.M., January 31. For additional information contact the College of Engineering, 255 West Engineering Building, University of Michigan, Ann Arbor, Michigan.

PERSONAL ITEMS

Professor D. J. Struik, Massachusetts Institute of Technology, has been nominated to an Extraordinary Professorship at the National University of Mexico and elected an Honorary Member of the Mexican Mathematical Society.

American University: Mrs. Lucille Reifman, Associate, George Washington Univer-

sity, has been appointed Instructor; Dr. W. F. Shenton, Chairman, Department of Mathematics, has retired; Professor J. H. Smith has been appointed Chairman.

Boston University: Dr. Philip Maker has been appointed Assistant Professor; Mr. Chacko Abraham, Bombay University, has been appointed Instructor; Assistant Professor Donald Blackett has been promoted to Associate Professor; Professor E. B. Mode has retired with the title Professor Emeritus; Professor G. E. Noether is on leave as a Fulbright Lecturer at Tübingen; Professor Francis Scheid is on leave at the National Bureau of Standards.

Canisius College: Dr. T. D. Pozniak, St. Joseph's College, has been appointed Assistant Professor; Mr. R. L. Uschold has been promoted to Assistant Professor.

Cornell University: Dr. R. G. Heyneman, Research Assistant, University of California, Berkeley, and Dr. Constantine Kassimatis, Lehigh University, have been appointed Instructors.

DePaul University: Mr. E. G. McNiel, Instructor in Business Mathematics at the University, has been appointed Instructor in Mathematics; Dr. W. F. Darsow, Dr. L. M. Wiener, and Dr. G. L. Weiss have been promoted to Assistant Professors; Dr. Mary Weiss has obtained a postdoctoral research fellowship at the University of Chicago for 1957–58.

Eastern Michigan College: Mr. James H. Northey has been appointed Assistant Professor; Mr. P. D. Drees has been appointed Instructor.

Gonzaga University: Mr. B. E. Brown, Field Engineer, General Electric Company, and Associate Professor Ruth E. Dillavou, Western Montana College of Education, have been appointed Lecturers.

Lehigh University: Dr. J. W. Wall, Jr., recently at the Institute for Advanced Study, has been appointed Assistant Professor; Mrs. Marguerite Gravez, Interpreter, French Embassy, has been appointed Instructor; Mr. E. K. Dorff, Mr. John Nassar, and Mr. F. C. Oglesby, Graduate Assistants at the University, have been promoted to Instructors.

Marquette University: Dr. Frank Wagner, Midwest Research Corporation, Kansas City, Missouri, has been appointed Assistant Professor; Mr. William Weideman, Graduate Student, University of Michigan, and Miss Sheila Conheady, Graduate Student at the University, have been appointed Instructors; Mr. Ernest Gloyd has been appointed Assistant Instructor; Rev. L. J. Heider, Assistant Professor, has been promoted to Associate Professor; Dr. Earl Swokowski and Mr. John Kelley have been promoted to Assistant Professors.

Miami University: Mr. R. F. DeMar, Graduate Assistant, University of Wisconsin, has been appointed Instructor; Mr. C. C. Crell and Mr. R. W. Emmert have been promoted to Assistant Professors; Assistant Professor T. C. Holyoke is on leave on a Postdoctoral National Science Foundation Fellowship at the University of California.

Montana State University: Assistant Professor W. R. Ballard, Air Force Institute of Technology, and Mr. H. E. Reinhardt, Teaching Fellow, University of Michigan, have been appointed Assistant Professors; Dr. F. H. Young, Senior Engineer, Autonetics, Downey, California, has been appointed Associate Professor; Assistant Professor Wolfgang Schmidt is on leave at the University of Vienna.

New York University, Institute of Mathematical Sciences: Assistant Professor J. T. Schwartz, Yale University, has been appointed Associate Professor; Associate Professors Harold Grad, P. D. Lax, and Louis Nirenberg have been promoted to Professors; Research Associate Professor J. B. Keller has been promoted to Research Professor; Dr. Cathleen S. Morawetz has been promoted to Associate Professor; Dr. D. S. Carter, Los Alamos Scientific Laboratories, New Mexico, Mr. S. R. Foguel, Yale University, Professor M. P. Gaffney, Jr., Northwestern University, Dr. S. R. Goldner, Stellenbosch University, Union of South Africa, Dr. R. K. Juberg, University of Minnesota, Dr. T. E. Nieminen, University of Helsinki, Professor Robert Osserman, Stanford University,

Dr. R. N. Pederson, University of Minnesota, Dr. Maurice Roseau, Poitiers University, France, and Dr. Leonard Topper, U. S. Atomic Energy Commission, have been appointed Temporary Members for the academic year 1957–58.

New York University, Washington Square College: Research Associate Professor Joseph Keller has been promoted to Research Professor; Research Assistant Professor Samuel Karp has been promoted to Research Associate Professor; Assistant Professor Warren Hirsch has been promoted to Associate Professor.

Northwestern University: Dr. R. R. Goldberg, Mathematician, Westinghouse Atomic Power Division, Pittsburgh, Pennsylvania, Mr. R. J. Driscoll, Research Assistant at the University, and Dr. Bruno Harris, Postdoctoral NSF Fellow, Yale University, have been appointed Instructors; Dr. M. P. Drazin, Research Fellow, Trinity College, Cambridge, England, has been appointed Visiting Lecturer; Dr. Edoardo Vesentini, Pisa University and the Politechnico di Milano, has been appointed Lecturer; Assistant Professors Meyer Dwass and Alex Rosenberg have been promoted to Associate Professors; Professor W. T. Reid is on leave for the academic year 1957–58 as Visiting Professor at the University of California; Assistant Professor Esther Seiden was on leave at the University of Wisconsin for the fall semester of 1957–58; Associate Professor M. A. Rosenlicht is on leave on a Guggenheim award.

Oklahoma State University: Assistant Professor J. W. Hamblen has been promoted to Associate Professor and Director of the Computing Center; Assistant Professor R. D. Morrison has been promoted to Associate Professor.

Oregon State College: Professor Cornelius Lanczos, Dublin Institute for Advanced Studies, has been appointed Visiting Professor; Professor Gustav Lochs, University of Innsbruck, has been appointed Visiting Associate Professor; Dr. Helmut Groemer, Postdoctoral Student, University of Vienna, has been appointed Instructor; Mr. R. R. Reynolds has been promoted to Assistant Professor; Professor Emeritus W. E. Milne is Visiting Professor at U. S. Naval Postgraduate School, Monterey, California, during 1957–58.

Pennsylvania State University: A new program for the degree of Master of Engineering has been inaugurated. This program permits people with an undergraduate engineering background to earn this degree off campus with no residence requirements. Professor B. H. Bissinger, Chairman of the Department of Mathematics of Lebanon Valley College, has been appointed to the staff of the University to teach mathematics in this program and is teaching in Harrisburg during 1957–58.

Purdue University: Associate Professor John Warfield, University of Illinois, and Dr. Annette Sinclair, Research Fellow, Harvard University, have been appointed Associate Professors; Dr. George Pedrick, University of Kansas, Dr. Lincoln Turner, Research Assistant at the University, and Dr. Norman Alling, Lecturer, Columbia University, have been appointed Assistant Professors; Mr. James Abbott, Teaching Assistant, University of Illinois, Mr. Irvin Lynn, University of Cincinnati, and Mr. Joseph Stefanac, Captain, U. S. Navy, have been appointed Instructors; Mr. W. R. Fuller, Mr. Frank Kozin, Dr. Judah Rosenblatt, and Dr. A. M. Yaqub have been promoted to Assistant Professors; Assistant Professors Melvin Henriksen, C. R. Hicks, G. T. Miller, and Stanley Reiter have been promoted to Associate Professors; Assistant Professor P. W. Overman has retired; Professor Arthur Rosenthal has retired with the title Professor Emeritus.

Randolph-Macon Woman's College: A Conference, Science in Perspective, was held at the College on November 12–14, 1957. The purposes of the conference were to interpret some of the major problems and issues before natural scientists today and to stimulate the interest of both secondary school and college students in the natural sciences. The following mathematicians participated in the program. Professor A. W. Tucker, Princeton University, spoke on *Natural Science and the Social Sciences* in the Symposium on Relationships between Natural Science and Other Fields; In the Sym-

posium on Current Issues and Developments, Dr. Louis Robinson, Manager of the Mathematics and Applications Department of the I.B.M. Corporation, spoke on *Applications of the IBM-704 Data-Processing Machine to the Computing Problems in Tracking the Earth Satellite;* Dean Mina Rees, Hunter College, gave an address entitled *Problems for Machines* in the Symposium on New Horizons in Science.

Rutgers University: Professor O. E. Villamayer, University of La Plata, Argentina, has been appointed Visiting Professor; Assistant Professor Harry Gonshor, Pennsylvania State University, has been appointed Assistant Professor; Mr. D. G. Malm, Teaching Assistant, Brown University, has been appointed Instructor; Dr. E. R. Gentile, Assistant, University of La Plata, has been appointed Lecturer; Assistant Professor V. L. Shapiro has been promoted to Associate Professor; Dr. R. E. Heaton and Dr. Abe Shenitzer have been promoted to Assistant Professors.

Stanford University: Dr. Karel deLeeuw, University of Wisconsin, has been appointed Assistant Professor; Assistant Professor G. E. Latta has been promoted to Associate Professor.

Syracuse University: Visiting Associate Professor Wolfgang Jurkat, Ohio State University, has been appointed Visiting Associate Professor; Assistant Professors Kathryn Morgan and O. O. Pardee have been promoted to Associate Professors.

Texas Christian University: Mrs. Florence Messinger has been appointed Instructor; Professor J. I. Tracy has retired.

University of Akron: Dr. Karl Johannes, Wisconsin State College, Whitewater, has been appointed Associate Professor; Miss Catharine Howard has been appointed Instructor; Assistant Professor Ernest Tabler has been promoted to Associate Professor; Associate Professor Louis Ross was Visiting Associate Professor at Western Reserve University during the summer of 1957. The University is now offering a Master of Science Degree in Engineering requiring 18 credit hours of graduate work in mathematics and physics. The University received a gift of an aerojet-general nucleonics Model 201 Nuclear Reactor from the General Tire and Rubber Company.

University of Alaska: Professor F. D. Parker, on leave from Clarkson College of Technology, is Professor and Head of the Department of Mathematics; Mr. W. A. Peterson, Boeing Aircraft Company, and Mrs. Ingrid Owren have been appointed Instructors; Mr. J. O. Distad has been promoted to Assistant Professor.

University of California, Berkeley: Professor Hene de Volgelaere, Notre Dame University, and Professor Istvan Fary, University of Montreal, have been appointed Visiting Associate Professors; Professors Leo Breiman and Steven Orey, University of Minnesota, have been appointed Visiting Assistant Professors; Dr. Jacob Feldman, National Science Foundation Fellow, Columbia University, Professor R. H. Lehman, and Dr. Maurice Sion, Member, Institute for Advanced Study, have been appointed Assistant Professors; Dr. A. H. Lightstone and Dr. Marianne Smith have been appointed Lecturers for 1957–58; Professors Harley Flanders and J. L. Kelley are on sabbatical leave in Cambridge, England; Miss Eva Kallin is spending the year in Cambridge, England, on an American Association of University Women Fellowship; Professor Albert Tarski has been appointed honorary member of the Dutch Mathematical Society; Professor Bernard Friedman, New York University, has been appointed Professor.

University of Delaware: Dr. C. C. Braunschweiger, Purdue University, has been appointed Assistant Professor; Dr. W. A. Thompson, Jr., Air Defense Board, has been appointed Associate Professor; Miss Edith A. McDougle has retired with the title Instructor Emeritus.

University of Detroit: Mr. J. B. Eckstein, Teacher, Harper Woods High School, Michigan, and Dr. Violet B. Haas, University of Connecticut, have been appointed Assistant Professors; Mr. R. G. Kane, Mr. G. E. Meike, and Miss Maryjo M. Nichols, have been appointed Instructors; Associate Professors G. E. Markle, E. D. McCarthy, and Emily C. Pixley have been promoted to Professors; Mr. J. G. Sowul has received a National

Science Foundation Faculty Fellowship and is studying at Wayne State University.

University of Kansas: Associate Professor Frank Gamblen, University of Western Australia, Nedlands, has been appointed Visiting Associate Professor; Dr. U. W. Hochstrasser, Assistant Professor at American University and Mathematician, Computation Laboratory, National Bureau of Standards, has been appointed Associate Professor and Director of the Computation Center; Dr. J. C. Lillo, Graduate Student, Princeton University, has been appointed Assistant Professor; Associate Professor George Springer has been promoted to Professor; Assistant Professor W. F. Donoghue, Jr., has been promoted to Associate Professor; Professor Nachman Aronszajn is on sabbatical leave in Cambridge, England, and Belgium; Professor Robert Schatten has received a research grant from the National Science Foundation and is at the University of California, Berkeley; Associate Professor W. F. Donoghue, Jr., presented a paper in Helsinki, Finland, at the International Colloquium on Function Theory, sponsored by the International Union of Mathematicians.

University of Maryland: Dr. G. J. Rieger, Privatdozent, University of Giessen, Germany, Professor John Horvath, University de los Andes, and Dr. J. A. Hummel, Postdoctoral Fellow, Stanford University, have been appointed Assistant Professors; Mrs. Shuh-yin Mar has been appointed Instructor; Dr. J. R. Mayor, American Association for the Advancement of Science, has been appointed Part-time Professor; Associate Professor G. S. S. Ludford is on sabbatical leave and is studying at Harvard University.

The Advisory Board on Education of the National Academy of Sciences has announced the first experimental use of color television in the teaching of a graduate course, Foundations of Analysis, offered by the University. The closed-circuit color T.V. facility at Walter Reed Army Medical Center is employed in teaching this course to a group of inservice high-school teachers of mathematics and science in the Washington, D. C. area. Professor R. A. Good of the University is lecturer for the course and Dr. J. R. Mayor, Director of Education, AAAS, is serving as consultant. A grant from the Fund for the Advancement of Education of the Ford Foundation has made possible the production of color kinescopes which will be used in the comparison of various techniques. These kinescopes will later be made available to other suitably equipped institutions for further evaluation tests.

University of Mississippi: Associate Professors N. A. Childress and R. D. Sheffield have been promoted to Professors; Professor Benjamin Ernest Mitchell has retired.

University of Missouri: Dr. D. F. Dawson, University of Texas, has been appointed Assistant Professor; Professor G. M. Ewing is on leave for the year 1957–58 as a mathematician at Fort Sill, Oklahoma.

University of Montreal: Dr. Alexis Zinger has been appointed Assistant Professor; Assistant Professors Geoffrey Fox and Jean Maranda have been promoted to Associate Professors; Professor Maurice L'Abbé has been appointed Chairman of the Department of Mathematics.

University of Rochester: Professor W. F. Eberlein, University of Wisconsin, has been appointed Professor; Mr. William Browder, Princeton University, Mr. P. J. Cohen, National Science Foundation Fellow, University of Chicago, Dr. Arshag Hajian, Research Assistant, Yale University, and Mr. D. E. Schroer, Acting Instructor, University of California, Berkeley, have been appointed Instructors; Associate Professors Dorothy L. Bernstein and Walter Rudin have been promoted to Professors; Professor Bernstein has received a two-year grant from the National Science Foundation and is now at the Institute for Numerical Analysis, University of California at Los Angeles; Dr. R. W. MacDowell is at Cornell University on a Science Faculty Fellowship from the National Science Foundation.

University of Virginia: Dr. R. T. Ives, University of Washington, has been appointed Instructor; Assistant Professor E. C. Paige, University of Illinois, has been appointed Assistant Professor.

University of Western Ontario: Mr. Douglas Clarke, Officer, R.C.A.F., and Mr. W. H. Adamson have been appointed Instructors.

Yale University: Dr. Leonard Gross, Research Assistant, University of Chicago, Dr. Walter Koppelman, Coder-Programmer, Institute of Mathematical Sciences, New York University, Dr. P. J. Message, Cambridge University, England, and Dr. A. B. Simon, Tulane University, have been appointed Instructors; Dr. George Seligman has been promoted to Assistant Professor.

Dr. W. R. Abel, University of Missouri, has been appointed Instructor at the University of Nebraska.

Dr. J. W. Addison, who has spent the past year at the Institute for Advanced Study and the Mathematical Institute of the Polish Academy of Sciences, has been appointed Assistant Professor at the University of Michigan.

Mr. C. E. Antle, Aerophysics Engineer, Convair, Fort Worth, Texas, has been appointed Instructor at Missouri School of Mines and Metallurgy.

Assistant Professor H. A. Arnold, University of California, Davis, has been promoted to Associate Professor.

Dr. D. D. Aufenkamp, Systems Laboratories Corporation, Sherman Oaks, California, has a position as Associate Research Scientist at Lockheed Missile Systems Division, Palo Alto, California.

Associate Professor Ray Authement, McNeese State College, has been appointed Associate Professor at Southwestern Louisiana Institute.

Associate Professor G. B. Ax, Virginia Military Institute, has been promoted to Professor.

Dr. W. L. Baily, Jr., Massachusetts Institute of Technology, has been appointed Assistant Professor at the University of Chicago.

Associate Professor Grace E. Bates, Mount Holyoke College, has been promoted to Professor and Chairman of the Department of Mathematics.

Mr. Jacques Bazinet, Graduate Student, University of Montreal, has been appointed Professor at the University of Sherbrooke, Province of Quebec, Canada.

Associate Professor R. F. Bell, Eastern Washington College of Education, has been promoted to Professor and Chairman of the Department of Mathematics.

Mr. W. C. Bennewitz, Graduate Assistant, University of Illinois, has been appointed Instructor at the University of Southern California.

Assistant Professor Donald C. Benson, Carnegie Institute of Technology, has been appointed Assistant Professor at the University of California, Davis.

Mrs. Shirley A. Blackett, Northeastern University, has been promoted to Assistant Professor.

Mr. D. J. Boyce, Graduate Assistant, Oklahoma State University, has been appointed Instructor, Central State College.

Associate Professor B. W. Brewer, Oregon State College, has been appointed Professor at Agricultural and Mechanical College of Texas.

Mr. C. A. Bridger, Director, Bureau of Vital Statistics, Division of Health, Jefferson City, Missouri, has been appointed Chief, Bureau of Statistics, State Department of Public Health, Springfield, Illinois.

Dr. F. A. Butter, Jr., Research Physicist, Hughes Aircraft Company, Culver City, California, has a position as an engineering specialist at Northrop Aircraft, Inc., Hawthorne, California.

Mr. A. J. Carlan, Research Physicist, American Optical Company, Southbridge, Massachusetts, is now employed as Physicist at Hoffman SemiConductor Division, Evanston, Illinois.

Miss L. Virginia Carlton, Visiting Lecturer, Northwestern University, has been appointed Chairman, Department of Mathematics, Centenary College.

Assistant Professor R. C. Carson, Lehigh University, has been appointed Assistant Professor at Western Reserve University.

Professor Abraham Charnes, Purdue University, has been appointed Research Professor, Northwestern University.

Miss Virginia B. Christian, Instructor, Eastern Illinois State College, has been appointed Instructor at South Dakota State College.

Assistant Professor R. A. Clark, Case Institute of Technology, has been promoted to Associate Professor.

Dr. M. L. Coffman, Senior Nuclear Engineer, Convair, Ft. Worth, Texas, has been appointed Associate Professor at Abilene Christian College.

Mr. Gerald Derman, Teaching Assistant, Rutgers University, is now a teacher at New Brunswick High School, New Jersey.

Dr. Jim Douglas, Jr., Assistant Research Engineer, Humble Oil and Refining Company, Houston, Texas, has been appointed Assistant Professor at Rice Institute.

Mr. R. E. Ekstrom, Mathematician, U. S. Naval Ordnance Plant, Indianapolis, Indiana, has been appointed Research Associate, Department of Engineering Mechanics, University of Florida.

Assistant Professor William Golomski, Marquette University, is now employed by the Oscar Meyer Company, Madison, Wisconsin.

Assistant Professor W. H. Greub, University of Maryland, has been appointed Assistant Professor at Johns Hopkins University.

Dr. J. K. Hale, Sandia Corporation, Albuquerque, New Mexico, has accepted a position as a member of the Mathematical Research Department, Remington Rand Univac, St. Paul, Minnesota.

Mr. R. W. Harruff, Student, Kent State University, is now Research Engineer at North American Aviation, Downey, California.

Associate Professor G. P. Henderson, University of Western Ontario, has joined the research staff of Imperial Oil, Ltd., Toronto, Ontario, Canada.

Mr. William Hoyt, Northwestern University, is an ONR Fellow at Johns Hopkins University.

Assistant Professor W. E. Jenner, Northwestern University, has been appointed Associate Professor at Bucknell University.

Professor Ralph Johanson, Boston University, has a position as Mathematician with the I.B.M. Corporation.

Mr. H. C. Kerr, Blinn College, is employed as Research Engineer with Convair, Daingerfield, Texas.

Mr. D. R. King, Teaching Assistant, Rutgers University, is now employed by the Radio Corporation of America, Camden, New Jersey.

Assistant Professor G. F. Leger, Syracuse University, has been appointed Assistant Professor at the University of Pittsburgh.

Research Associate Professor S. C. Lowell, New York University, has been appointed Professor and Director of the Graduate Program in Applied Mathematics and Physics, Adelphi College.

Mr. P. T. Mielke, Senior Group Engineer, Boeing Airplane Company, Seattle, Washington, has been appointed Associate Professor at Wabash College.

Dr. C. N. Mills, Visiting Lecturer, Florida State University, has been appointed Professor at Sioux Falls College.

Dr. H. W. Milnes, Senior Mathematician, Data Processing Group, General Motors Corporation, Detroit, Michigan, is now Senior Research Scientist, Science Group of the Corporation.

Dr. Josephine Mitchell, Westinghouse Research Laboratory, has been appointed Associate Professor at the University of Pittsburgh.

Dr. A. E. Nussbaum, Rensselaer Polytechnic Institute, has been promoted to Assistant Professor.

Mr. J. M. Osborn, Jr., Yale University, has been appointed Assistant Professor, Georgia Institute of Technology.

Dr. M. H. Pearl, University of Rochester, has a position with the National Bureau of Standards, Commerce Department, Washington, D. C.

Assistant Professor O. P. Sanders, University of Arkansas, has been appointed Professor at Louisiana Polytechnic Institute.

Dr. Albert Schild, Temple University, has been promoted to Associate Professor.

Professor J. P. Scholz, Head, Department of Mathematics, Western College for Women, has been appointed Professor at New Mexico Institute of Mining and Technology.

Mr. Aaron Siegel, Teaching Assistant, Rutgers University, has been appointed Instructor at the College of South Jersey of the University.

Professor Jack Silber, Roosevelt University, has returned to the University after spending four months as Consultant to the Operations Analysis Office at the Air Force Missile Test Center.

Assistant Professor G. L. Spencer, University of Maryland, has been appointed Associate Professor at Williams College.

Mr. Frederick R. White, University of Buffalo, has a position as Mathematician, Sylvania Electric Products, Mountain View, California.

Dr. J. B. Wilson, Assistant, University of Florida, has been appointed Assistant Professor at North Carolina State College.

Mr. A. W. Yonda, Temple University, has been appointed Associate Scientist at AVCO Manufacturing Company, Lawrence, Massachusetts.

Mr. Joseph Zilber, Visiting Lecturer, Northwestern University, is now Editorial Consultant, *Mathematical Reviews*, American Mathematical Society, Providence, Rhode Island.

Assistant Professor Stanley Bolks, Purdue University, died on January 26, 1957.

Associate Professor Mildred Crawford, Eastern Michigan College, died in November, 1956.

Professor Emeritus C. B. Upton, Teachers College, Columbia University, died on September 25, 1957.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

TELEVISION AND FILMS

It is proposed to collect in one issue of the MONTHLY as many articles as possible on the use of television and of films in the teaching of mathematics. Articles intended for this special issue should be submitted to the Editor, Professor R. D. James, Department of Mathematics, University of British Columbia, Vancouver 8, Canada, as soon as possible and in any event, by March 15, 1958. Since space is limited, such articles should be concise and should not include unnecessarily elaborate statistical data.

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 160 persons have been elected to membership by the Board of Governors on applications duly certified.

JOHN ABRAMOWICH, M.A. (Toronto) Res. Asst., University of California, Berkeley.

WILFREDO ACARÓN, B.A. (Interamerican) Teacher, Cabo Rojo High School, Puerto Rico.

GEORGE ALEXANDER, JR., B.A. (Baylor) Math. Phillips Petroleum Co., Waco, Texas.

SAM ALEXANDER, M.S. (C.C.N.Y.) Teacher, Bronx High School of Science.

KENNETH I. APPEL, M.A. (Michigan) Teaching Fellow, University of Michigan.

ALBERT ARCESE, B.S. (Northeastern) Math., Radio Corp. of America, Waltham, Mass.

MRS. BLANCHE C. BADGER, Ph.D. (George Peabody) Chm., Dept. of Math., Longwood College.

RAYMOND BAGLEY, Student, Whittier College.

FOSTER BAKER, B.S. (Northwest Missouri S.C.) Teacher, Nortonville Rural High School, Kan.

CLYDE E. BARFIELD, B.S. in M.E. (West Coast) Design Engr., Aerojet General Corp., Azusa, Calif.

VICTOR W. BAUMAN, M.S. (Colorado) Asst. Prof., Colorado School of Mines.

HERMAN J. BIESTERFELDT, JR., B.S. (Pennsylvania S.U.) Grad. Asst., Physics, Pennsylvania State University.

MOSES A. E. BLAIR, B.S. (Johnson C. Smith) Teacher, Highland High School, Gastonia, N. C.

ARDEL J. BOES, Student, St. Ambrose College.

HENRY G. BRAY, B.A. (San Diego S.C.) Grad. Asst., Iowa State College.

KENNETH A. BRONS, Ph.D. (Illinois) Applied Science Representative, I.B.M. Corp., River Forest, Ill.

CHARLES M. BRUEN, B.A. (Illinois) Math.-Programmer, I.B.M. Corp., Endicott, N. Y.

RONALD E. BULLOCK, B.S. (Louisiana Poly. Inst.) Nuclear Engr., Convair, Fort Worth, Texas.

WINIFRED K. BURROUGHS, M.A. (Radcliffe) Instr., Ohio Wesleyan University.

MRS. CONCEPCIÓN R. CANTRELL, B.S. (Puerto Rico) Teacher, Corozal High School, Puerto Rico.

PAUL G. CAUGHRAN, B.S. in Ed. (Southwest Missouri S.C.) Springfield, Mo.

JOYCE C. M. CIMELUS, M.S. (Purdue) Instr., Butler University.

CLINTON L. CONNER, M.S. in E.E. (Colorado) Asst. to the Dean, College of Engineering, University of Colorado.

MICHAEL J. CONNOLLY, B.A. (Brandeis) Field Representative, Addison-Wesley Publishing Co., Reading, Mass.

JAMES E. COOK, Student, Simpson College.

ACHILLES W. COUTRIS, M.A. (Columbia) Grad. Student, Columbia University.

LONNIE CROSS, Ph.D. (Cornell) Asso. Professor, Atlanta University.

WILLIAM F. CULLITON, B.S. (Canisius) Programmer, Metals Research Labs., Niagara Falls, N. Y.

JAMES W. DAILEY, A.M. (Boston C.) Teacher, Boston Technical High School; Consultant, Arthur D. Little.

WILLIAM C. DAVIS, B.S. (Texas) Aeronautical Res. Administrator, Army Ballistic Missile Agency, Huntsville, Ala.

MAX A. DENGLER, Ph.D. (Vienna) Adviser, AiResearch of Arizona, Phoenix, Ariz.

BENJAMIN A. DENT, Student, Haverford College.

WILLIAM R. DERRICK, Student, Oklahoma State University.

DANIEL A. DESALVO, Product Researcher, Eaton Manufacturing Co., South Euclid, Ohio.

REV. BRO. ARTHUR T. DEVLIN, M.S. (St. John's) Teacher, Cardinal Hayes High School, Marian College, New York, N. Y.

CARMEN D. DÍAZ, B.A. (Puerto Rico) Teacher, Guillermo Estevez High School, Naranjito, Puerto Rico.

ROBERTO DIAZ-FERNÁNDEZ, Ph.D. (California, Berkeley) Asst. Professor, University of Puerto Rico.

WILLIAM P. DRISCOLL, B.A. (St. Mary's) Ltjg., U. S. Navy, Little Creek, Va.

DONALD V. EASTER, M.S. (Florida S.U.) Staff Member, Sandia Corp., Livermore, Calif.

MRS. HENRIETTE B. ENGEL, M.A. (Columbia)

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BASIL M. FEDOROVSKY, M.S. (Naval Acad., Russia) Asst. Professor, Merrimack College.

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HARRY J. FIGGE, President, Harry J. Figge and Associates, Des Moines, Ia.

WILLIAM T. FLETCHER, M.S. (North Carolina C.) Graduate Student, North Carolina College at Durham.

GEORGE FORESTER, Student, Reed College.

GEORGE K. FRANCIS, Student, University of Notre Dame.

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MRS. ROSÁRIO GONZÁLEZ DE ROLDÁN, B.A. (Interamerican) Teacher, Aquadilla High School, Puerto Rico.

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BURLEY D. HAM, Akron, Ohio.

MR. HAMEEDULLAH, M.A. (Panjab U.) Lecturer, Talim-ul-Islam College, Rabwah, West Pakistan.

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RONALD J. NUNKE, Ph.D. (Chicago) Instr., Yale University.

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MRS. AURORA ORTIZ DE SEGUI, B.A. (Puerto Rico) Teacher, Dept. of Instruction, Hoto Rey, Puerto Rico.

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FRANCIS S. PERRYMAN, B.S. (London) Actuary and Asst. U. S. Manager, Royal-Globe Insurance Group, Scarsdale, N. Y.

CARLOS R. PETIT, Engg. (Argentine) Translator, RCA International Div., Clark, N. J.

JOSE F. PITA DE MACEDO, Licentiate (Lisbon) Institute of High Culture, Lisbon, Portugal.

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ROBERT A. RAGAZZO, B.S. (Manhattan) Math., Griffiss Air Force Base, Rome, N. Y.

BURTON RANDOL, Student, Rice Institute.

LAWRENCE C. REDMOND, Student, Illinois Institute of Technology; Asso. Design Engr., American Machine & Foundry Co., Chicago, Ill.

MARGARET V. RHOADS, Ed.D. (Columbia) Professor, State Teachers College, Slippery Rock, Pa.

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REV. CHARLES K. RUSTERHOLTZ, M.S. (Notre Dame) Instr., Bellarmine College.

MRS. MARY E. SACCOMANNO, M.S. (De Paul) Instr., Wright Junior College.

LEONARD G. SALVIN, Student, Clark University; Statistician, Worcester Foundation for Experimental Biology, Shrewsbury, Mass.

BENJAMIN SAPOLSKY, M.S. (Pennsylvania) Teacher, Olney High School, Philadelphia, Pa.

SAMUEL S. SASLAW, Ph.D. (M.I.T.) Professor, U. S. Naval Academy

MARGARET A. SASS, A.B. (Dunbarton C.) Math., Naval Research Lab., Washington, D. C.

BROTHER THOMAS A. SCHOEN, S.M., B.S. (Dayton) Grad. Student, University of Cincinnati.

DONALD L. SHELL, M.S. (Cincinnati) Computations Consulting Specialist, General Electric Co., Cincinnati, Ohio.

SISTER M. KENNETH KOLMER, Ad.P.P.S., B.S. (St. Louis) Grad. Student, St. Louis University.

SISTER MARIE JESSE MORRISON, Ph.D. (Catholic) Asst. Professor, Nazareth College, Mich.

WILLIAM T. SLEDD, III, A.B. (Murray S.C.) Instr., University of Kentucky.

LANCE SMALL, Student, Roosevelt High School, Yonkers, N. Y.

PALMER F. SMITH, B.S. (Colorado) Instr., University of Colorado.

WILLIAM K. SMITH, Ph.D. (Michigan) Asso. Professor, Antioch College.

CLIFFORD SPECTOR, Ph.D. (Wisconsin) Asst. Professor, Ohio State University.

ALEXANDER G. STONE, LL.D. (Debrecen, Hungary) Math., Johns Hopkins University, Applied Physics Lab.

ARTHUR L. STONE, Student, Reed College.

BERKLEY A. TAGUE, Student, Wesleyan University.

PATRICIA A. TUCKER, Student, Kansas State College.

HERBERT P. VALKER, M.A. (Michigan) Actuarial Asst., A. S. Hansen, Consulting Actuaries, Oak Park, Ill.

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SARAH VELÁZQUEZ MARRERO, B.A. (Sacred Heart) Teacher, Utuado's High School, Puerto Rico.

MURRAY F. WARD, B.S. (Acadia) Teacher, New Germany Rural High School, Nova Scotia.

WILLIAM F. WHITMORE, Ph.D. (California, Berkeley) Chief Scientist, Special Projects Office, Bureau of Ordnance, Washington, D. C.

MRS. LOUISE C. WILLIAMS, B.S. (Misericordia) Instr., Baltimore County Board of Education, Md.

JEAN M. WRIGHT, A.B. (Spelman) Teacher, Spelman College.

GEORGE G. YOUNG, M.Ed. (Alabama S.C.) Teacher, Yazoo City Training School, Miss.

A SPECIAL MEETING OF THE IOWA SECTION

The first meeting of the Iowa Section of the Mathematical Association of America with a sizeable group of high school teachers of mathematics was held at the State University of Iowa, Iowa City, Iowa, on October 10, 1957, in connection with the Twenty-seventh Annual Conference of Teachers of Mathematics. Provost H. H. Davis of the

State University of Iowa (a former teacher of high school mathematics) welcomed the participants with *Some observations concerning mathematics.* Professors M. F. Smiley and H. V. Price presided at the morning and afternoon sessions, respectively. There were 98 persons in attendance of which 22 were members of the Iowa Section.

The program consisted of four invited lectures. At the morning session, Professor Henry Van Engen, Iowa State Teachers College, spoke on *Proposals of the Commission on Mathematics of the College Entrance Examination Board* and Professor H. P. Evans, University of Wisconsin, spoke on *The need for better coordination between high school and college mathematics.* The speakers of the afternoon session were Professor Max Beberman, University of Illinois, *The University of Illinois School Mathematics Project,* and Mr. Frank Allen, Lyons Township High School and Junior College, LaGrange, Illinois, *The work of the Secondary School Curriculum Committee of the National Council of Teachers of Mathematics.* These addresses elicited lively discussion by both college and high school teachers of mathematics.

M. F. SMILEY, *Governor*

CALENDAR OF FUTURE MEETINGS

Thirty-ninth Summer Meeting, Massachusetts Institute of Technology, Cambridge, Massachusetts, August 25–28, 1958.

Forty-second Annual Meeting, University of Pennsylvania, Philadelphia, Pennsylvania, January 22–23, 1959.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Washington and Jefferson College, Washington, Pennsylvania, May 3, 1958.

ILLINOIS, Illinois College, Jacksonville, May 9–10, 1958.

INDIANA, May 13, 1958.

IOWA, Drake University, Des Moines, April 18, 1958.

KANSAS

KENTUCKY, University of Kentucky, Lexington, April 19, 1958.

LOUISIANA-MISSISSIPPI, Loyola University, New Orleans, February 21–22, 1958.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Randolph-Macon Woman's College, Lynchburg, Virginia, April 26, 1958

METROPOLITAN NEW YORK, Hofstra College, Hempstead, New York, April 19, 1958.

MICHIGAN, University of Michigan, Ann Arbor, March 22, 1958.

MINNESOTA

MISSOURI, University of Missouri, Columbia, April 26, 1958.

NEBRASKA, University of Nebraska, Lincoln, April 19, 1958.

NEW JERSEY, Rutgers University, New Brunswick, November 1, 1958.

NORTHEASTERN

NORTHERN CALIFORNIA, San Francisco State College, January 18, 1958.

OHIO, Denison University, Granville, April 26, 1958.

OKLAHOMA

PACIFIC NORTHWEST, Oregon State College, Corvallis, June 20, 1958.

PHILADELPHIA

ROCKY MOUNTAIN, Colorado State College, Greeley, May 9–10, 1958.

SOUTHEASTERN, University of Florida, Gainesville, March 14–15, 1958.

SOUTHERN CALIFORNIA, Pasadena City College, March 8, 1958.

SOUTHWESTERN, University of New Mexico, Albuquerque, April 11–12, 1958.

TEXAS, Baylor University, Waco, April, 1958.

UPPER NEW YORK STATE, Ecole Polytechnique and University of Montreal, Montreal, Quebec, Canada, May, 1958.

WISCONSIN, Carroll College, Waukesha, May, 1958.

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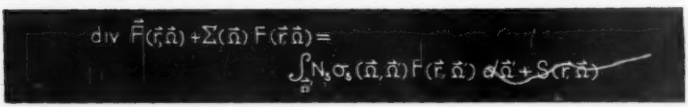
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